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Towards a Dimension Formula for Automorphic Forms on $O(\mathrm{II}_{2,10})$

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Für meine Eltern.

Zusammenfassung

Diese Dissertation beschäftigt sich mit der Bestimmung von Dimensionsformeln für spezielle orthogonale Modulformen, die mit dem $II_{2,10}$-Gitter in Zusammenhang stehen.

Für eine vorgegebene arithmetische Gruppe ist die Dimension der Räume dieser orthogonalen Modulformen ein Polynom zehnten Grades im Gewicht. Durch Nutzung des Hirzebruch–Riemann–Roch-Theorems und Hirzebruch–Mumford-Proportionalität lässt sich dieses bis auf einen geometrischen Fehlerterm exakt bestimmen; der Fehlerterm ist ein lineares Polynom, dessen Koeffizienten durch Schnittprodukte toroidaler Randdivisoren und bestimmter logarithmischer Chernklassen gegeben sind.

In dieser Arbeit beschreiben wir diesen Fehlerterm genauer und bestimmen wichtige Bestandteile. Hierfür konstruieren wir eine spezielle toroidale Kompaktifizierung der zum $II_{2,10}(N)$-Gitter assoziierten orthogonalen Modulvarietät und untersuchen deren Geometrie. Wir beschreiben zudem einen wesentlichen Teil der Schnitttheorie ebendieser Kompaktifizierung und reduzieren damit die Berechnung des linearen Koeffizienten des Fehlerterms auf ein kombinatorisches Problem. Schließlich geben wir Methoden an, welche die Berechnung des konstanten Koeffizienten des Fehlerterms ebenfalls auf kombinatorische Probleme reduzieren; inbesondere können wir eine Darstellung des Fehlerterms ohne logarithmische Chernklassen formulieren.

Acknowledgements

A thesis like this, or a scientific work in general, usually leaves not much space for personal matters. Nonetheless, I think these few introductory pages, before the real work for the motivated and interested reader even begins[1], are the appropriate place to say a heartfelt 'thank you' to some people. Undoubtedly, I will forget to mention people who contributed to this work and I'm sorry for this. Please be sure that I do appreciate it, even though I might not be aware while typing this.

First and foremost, I'd like to thank my family and friends for always supporting me, regardless of the level of frustration, anger or enthusiasm I was suffering from. I am deeply grateful for tolerating my sometimes tiring behavior. Also, I'd like to thank my bear[2] friend Fred Bär for his useful comments and suggestions as well as for his emotional support under any circumstances.

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[1] I've done my share of the work, it's your turn now
[2] not a typo
[3] unless I find someone else to blame
[4] we are not in accordance about the ratio

I disagree strongly with whatever work this quote is attached to.

Randall Munroe

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Introduction

In this thesis we provide some of the foundations necessary to prove a dimension formula for automorphic forms on $\mathrm{O}(\mathit{II}_{2,10})$. In particular, we describe a smooth compactification of the orthogonal modular variety corresponding to a neat discriminant kernel $\widetilde{\mathrm{SO}}^+(\mathit{II}_{2,10}(N)) \subseteq \mathrm{O}^+(\mathit{II}_{2,10})$ and study its geometry. This allows us to reduce the problem of determining the dimension formula for such a group to the calculation of intersection products without logarithmic factors.

The overall strategy of this approach is strongly influenced by the work of Andrew Fiori who devised many of the tools used here.

Orthogonal modular forms

Let L be an even non-degenerate lattice of signature $(2, n)$. The quotient $\mathcal{D} = \mathcal{D}_L$ of the real Lie group $\mathrm{O}^+(L \otimes \mathbb{R})$ by one of its maximal compact subgroups is an *orthogonal symmetric domain*, a Hermitian manifold with additional structure. A model of this space is any one of the two connected components of

$$\mathcal{K}^{\pm} = \left\{ [Z] \in \mathbb{P}(L \otimes \mathbb{C}) \,|\, ([Z], [Z]) = 0 \text{ and } ([Z], \overline{[Z]}) > 0 \right\}.$$

An orthogonal modular variety is the quotient $X(\Gamma)$ of $\mathcal{D}$ by an arithmetic subgroup $\Gamma \subseteq \mathrm{O}^+(L)$. This has the even richer structure of a *locally symmetric space of orthogonal type* and can be considered as an orthogonal Shimura variety. For certain choices of Γ it is a smooth projective complex variety of dimension n.

Spaces of this form are abundant in algebraic geometry and number theory due their prominent appearance in the theory of moduli spaces. Examples include the moduli space of polarized $K3$ surfaces of degree $2d$ (with $L = \mathit{II}_{2,18} \oplus \langle -2d \rangle$ and Γ its discriminant kernel) and the moduli space of Enriques surfaces (open subset of $X(L)$ with $L = \mathit{II}_{2,10}$ and Γ its discriminant kernel).

One method to study these varieties is via their k-pluricanonical forms, that is, by global sections of the k-fold tensor product of the canonical bundle $\Omega^n_{X(\Gamma)}$: The dimension of these spaces for varying k can be used to define the Kodaira dimension, a birational invariant of the variety $X(\Gamma)$.

These k-pluricanonical forms define meromorphic maps F from the affine cone $\tilde{\mathcal{K}}^+$ over $\mathcal{K}^+$ to the complex numbers with

i) $F(tZ) = t^{-nk} F(Z)$ for every $t \in \mathbb{C}^*$ and

ii) $F(\gamma Z) = F(Z)$ for all $\gamma \in \Gamma$

and a certain vanishing property at the boundary of $\tilde{\mathcal{K}}^+$.

Such functions are called *orthogonal cusp forms* and a slight generalization (by omitting the vanishing property) yields *orthogonal modular forms.* For fixed $k \in \mathbb{Z}$ and group Γ, the complex vector space $M_{nk}(\Gamma)$ of orthogonal modular forms is finite-dimensional.

An example of orthogonal modular forms is Borcherds' Φ_{12}-function. In a suitable coordinate system, it can be written as the infinite product

$$\Psi_{12}(Z) = e((\varrho, Z)) \prod_{\lambda \in II_{1,25}^+} (1 - e((\lambda, Z)))^{p_{24}(1-q(\lambda))}$$

with

$$\Delta^{-1}(\tau) = \sum_{n \geq 0} p_{24}(n) q^{n-1}$$

the Fourier expansion of the inverse of the Ramanujan Delta function Δ, the unique unimodular lattice $II_{1,25}$ of signature $(1, 25)$, and ρ a certain vector of $II_{1,25}$. This is a special case of *Borcherds' products*, a well-studied class of orthogonal modular forms.

As with the orthogonal modular varieties, the concept of orthogonal modular forms arises in a plethora of contexts, for example in fields as varied as the theory of Kac-Moody algebras or enumerative geometry; the latter due to the fact that any orthogonal modular form comes with certain Fourier series whose coefficients represent arithmetic information.

This diverse habitat of orthogonal modular forms is not a singular and exceptional situation: Orthogonal modular forms are only a (albeit very rich) subfield of the general theory of automorphic forms which are one of the three central components at the heart of the Langlands program which seeks to find a unified way of treating automorphic forms, the representation theory of algebraic groups and Galois groups.

Central problem: dimension formulas

Let L be a fixed lattice of signature $(2, n)$ and $\Gamma \subseteq \mathrm{O}^+(L)$ a finite-index subgroup. The dimensions of the spaces of orthogonal modular forms on $X(\Gamma)$ yield interesting arithmetic information on this orthogonal modular variety, so it is natural to ask the following question:

What is the dimension of $M_{nk}(\Gamma)$ and how does it depend on the weight $k \in \mathbb{Z}, k \geq 1$?

Ultimately, one would like to answer this in particular for the level-one case of $\Gamma = \mathrm{O}^+(L)$. This is the central question this thesis seeks to answer.

In general, one has to distinguish three different approaches for answering this question:

- exceptional maps
- trace formulas
- Riemann-Roch-type theorems

The conceptually easiest of those is the use of exceptional maps to identify automorphic forms and their domains with objects appearing naturally in contexts different from locally symmetric spaces, enabling the use additional techniques from the new mathematical environment.

Trace formulas are the natural tools from the point of view of representation theory and the Langlands program: They relate spectral data of algebraic groups to their geometry. There is a representation-theoretic reformulation of automorphic forms as certain square-integrable functions and one can consider the decomposition of the space of these functions into isomorphism classes of certain irreducible modules with multiplicities. These multiplicities can be interpreted as the spectral data of a trace formula, whose geometric data can be computed effectively.

In this work, we will take the third strategy and use Riemann-Roch-type theorems: The idea behind this approach is to interpret modular forms as global sections of a suitable vector bundle $\mathcal{V}$ on $X(\Gamma)$, so

$$M_{nk} = H^0(X(\Gamma), \mathcal{V}),$$

and extract the dimension of the space in question from the *Euler characteristic*

$$\chi(\mathcal{V}) = \sum_{i=0}^{\infty} \dim H^i(X(\Gamma), \mathcal{V})$$

by the use of vanishing theorems that ensure the triviality of the higher cohomology groups. The Hirzebruch-Riemann-Roch theorem then expresses this Euler characteristic in terms of intersection products of certain characteristic classes of $\mathcal{V}$ and the tangent bundle $\mathcal{T}_{X(\Gamma)}$ of $X(\Gamma)$, the *Chern classes*.

These approaches have been employed in various situations:

- Hashimoto and Ueda in [HU14] construct an exceptional bimeromorphic map

  $$\mathcal{D}_{II_{2,10}}/\mathrm{O}^+(II_{2,10}) \dashrightarrow \mathbb{P}(4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42)$$

  from the orthogonal modular variety corresponding to $II_{2,10}$ to a weighted projective space that is an isomorphism in codimension 1. This can be used to describe the corresponding ring of modular forms in terms of generators and relations; it is easy to extract a dimension formula from this.

- The work of Taibi in [Tai17] applies trace formulas to the dimension problem of Siegel modular forms. His result is a rather complicated but explicit formula describing the dimension of the space of Siegel cusp forms of degree up to 7 and sufficiently high weights in terms of traces of families of cyclotomic polynomials.

- Tsushima follows the Hirzebruch-Riemann-Roch (HRR) approach to the computation of dimension formulas for Siegel cusp forms in [Tsu80]. His work yields a dimension formula for higher level Siegel cusp forms on the Siegel upper half-plane of degree two and three. Later work in [Tsu82] generalizes this to dimension formulas for forms of degree two and of trivial level by the use of the holomorphic Lefschetz formula.

We explain the last approach and its specific problems in detail for the orthogonal case.

The HRR-approach to dimension formulas in the orthogonal case

The main problem with the HRR-approach is rooted in the strong assumptions for the applicability of the Hirzebruch-Riemann-Roch theorem: It applies only to smooth and projective varieties.

These are tough obstacles in many situations, including the ones we are interested in: For a lattice L of signature $(2, n)$ with $n \geq 3$ the orthogonal modular varieties $X(\Gamma)$ are non-compact for any choice of $\Gamma \subseteq \mathrm{O}^+(L)$; moreover, the orthogonal modular variety $X(\Gamma)$ may suffer from finite-quotient singularities due to torsion in the subgroup Γ.

The remedy for this is the restriction to smaller $\Gamma \subseteq \mathrm{O}^+(L)$, the *neat* subgroups which yield smooth modular varieties, and the use of suitable compactification theories which are compatible with the HRR-approach. Additionally, the standard use of vanishing theorems enforces the restriction to the case of orthogonal cusp forms.

A sensible roadmap to the computation of dimension formulas for $X(\mathrm{O}^+(I\!I_{2,10}))$ may therefore consist of the following steps:

(1) Compute dimension formulas for orthogonal cusp forms $S_{nk}(\Gamma)$ for neat and normal Γ via the HRR-approach by using suitable compactification theories.

(2) Extend the dimension formulas from orthogonal cusp forms to general orthogonal modular forms.

(3) Extend the dimension formulas from neat subgroups to the level-one case.

This is the approach taken by Tsushima in his work [Tsu80, Tsu82] on Siegel modular forms.

Step (1) is usually carried out by the use of Hirzebruch-Mumford proportionality and the theory of toroidal compactifications, step (2) may be approached via the combinatorics of suitable Eisenstein series and step (3) may be carried out by the use of fixed point theorems (holomorphic Lefschetz formula resp. Atiyah-Bott fixed point theorem) for the action of the finite group $\mathrm{O}^+(L)/\Gamma$.

This thesis will work out parts of step (1) for the unique even unimodular lattice $I\!I_{2,10}$ of signature $(2, 10)$ and neat subgroups Γ that arise as the discriminant kernels of rescalings $I\!I_{2,10}(N)$.

Mumford's strategy

We explain Mumford's approach to step (1) in more detail: Let L be again an even non-degenerate lattice of signature $(2, n)$ and $\Gamma \subseteq \mathrm{O}^+(L)$.

Assuming Γ to be neat prevents the problem of singularities on $X(\Gamma)$, while the non-compactness prevails, so one has to resort to the use of compactifications compatible with the HRR-approach to dimension formulas.

An orthogonal symmetric domain $\mathcal{D}$, corresponding to L as before, comes with a system of natural realizations indexed by rational boundary components $\mathcal{F}$, which are in correspondence with rational maximal parabolic subgroups of the isometry group $I(\mathcal{D})$: For a zero-dimensional $\mathcal{F}$ this realization is of the form

$$\mathcal{D} \cong \{z \in \mathbb{C}^n | \operatorname{Im} z \in \mathcal{C}(\mathcal{F})\}$$

with a certain real cone $\mathcal{C}(\mathcal{F})$ inside a Lorentzian space, so $\mathcal{D}$ is a tube domain. These domains are good models to see and repair the non-compactness of $\Gamma \backslash \mathcal{D}$: Taking the quotient by Γ shows that $X(\Gamma)$ looks locally like an open torus $(\mathbb{C}^*)^n$ and a compactification should fill in the missing points in this picture, which corresponds to adding points at the infinity of the cone $\mathcal{C}(\mathcal{F})$.

The Baily-Borel compactification simply takes the one-point compactification of each of these cones, and one can glue the resulting spaces over their common intersection to get the *Baily-Borel compactification* of $X(\Gamma)$, which is a compact variety, but usually very singular, rendering an application of a HRR-type theorem impossible.

The theory of toroidal compactification does better at this point: It uses toric geometry to embed the open tori into toric varieties. This can be pictured as adding arrangements of points at the end of the cone $\mathcal{C}(\mathcal{F})$. Any one of these completed cones gives a *partial compactification* of $X(\Gamma)$. If the chosen partial compactifications are especially well-behaved, their collection can be glued to yield an actual compactification of $X(\Gamma)$, which is again a complete algebraic variety. The exact construction is very involved and will occupy several chapters of this thesis.

A central feature of this construction is that the choice of the partial compactifications is by no means unique. This choice is usually expressed by the notion of Γ-*admissible families*. It is a collection $\Sigma = \{\Sigma(\mathcal{F})\}_{\mathcal{F}}$ of collections $\Sigma(\mathcal{F})$ of polyhedral cones, one for each zero-dimensional rational boundary component $\mathcal{F}$, such that:

1. Any collection $\Sigma(\mathcal{F})$ decomposes $\overline{\mathcal{C}(\mathcal{F})}^{\mathrm{rat}}$, a certain closure of $\mathcal{C}(\mathcal{F})$.
2. Any collection $\Sigma(\mathcal{F})$ is invariant under the action of a certain subgroup $\Gamma(\mathcal{F})$ of $\operatorname{Aut}(\mathcal{C}(\mathcal{F}))$ and has finitely many orbits.
3. The collection Σ is compatible with the action of Γ.

Mumford et al. in [AMRT10] and [Mum77] show that such families exist and the corresponding compactifications, usually denoted by $\overline{X}_{\Sigma}^{\mathrm{tor}}$, behave well in many regards: Suitable choices of Γ-admissible families lead to smooth projective varieties and vector bundles on $X(\Gamma)$ extend naturally to the toroidal compactifications $\overline{X}_{\Sigma}^{\mathrm{tor}}$; this amounts to compatibility with the HRR-approach. Moreover, the Chern numbers of the extended bundles are globally proportional to the Chern numbers of the compact dual $\check{\mathcal{D}}$ of the symmetric space $\mathcal{D}$.

This finally allows an application of the Hirzebruch-Riemann-Roch theorem: Mumford identifies the extension to $\overline{X}_{\Sigma}^{\mathrm{tor}}$ of the bundle of orthogonal cusp forms on $X(\Gamma)$ as the logarithmic cotangent bundle $\Omega^1_{\overline{X}^{\mathrm{tor}}_\Sigma}(\log\Delta)$ which can be defined via the short exact sequence

$$0 \longrightarrow \Omega^1_{\overline{X}^{\mathrm{tor}}_\Sigma} \longrightarrow \Omega^1_{\overline{X}^{\mathrm{tor}}_\Sigma}(\log\Delta) \longrightarrow \bigoplus_j \left(i_j\right)_* \mathcal{O}_{D_j} \longrightarrow 0 \ .$$

Here $\Delta = \overline{X}^{\mathrm{tor}}_\Sigma \setminus X = \{D_j |\, D_j \text{ irreducible component}\}$ is the compactification divisor. Using this, Mumford combines the Hirzebruch-Riemann-Roch theory on the smooth and compact $\overline{X}^{\mathrm{tor}}_\Sigma$ with the aforementioned proportionality result to prove the following approximate dimension formula:

Theorem (Mumford). For $k \geq 2$ we have

$$\dim(S_{nk}(\Gamma)) = \mathrm{Vol}_{\mathrm{HM}}(\Gamma)\mathcal{P}(k-1) + E(k)$$

with a polynomial $E(k)$ of degree 1, the Hilbert polynomial

$$\mathcal{P}(k) = \chi\left(\left(\Omega^n_{\check{\mathcal{D}}}\right)^{-k}\right)$$

of the compact dual $\check{\mathcal{D}} = \mathrm{SO}(2+n)/\left(\mathrm{SO}(2)\times\mathrm{SO}(n)\right)$ of $\mathcal{D}$ and the proportionality constant $\mathrm{Vol}_{\mathrm{HM}}(\Gamma) \in \mathbb{Q}$.

Note that Mumford's results are more general than the form we just presented: They apply to general locally symmetric spaces and their toroidal compactification, not necessarily originating from a lattice with signature $(2,n)$ as in our case.

This partly solves the problem of computing dimension formulas: The canonical part $\dim(S_{nk}(\Gamma)) - E(k)$ can be easily computed. Unfortunately, the computation of the error term depends very heavily on the exact choice of the toroidal compactification. In general, this consists of sums of intersection products of the Chern classes

$$c_i\left(\Omega^1_{\overline{X}^{\mathrm{tor}}_\Sigma}\right), c_j\left(\Omega^1_{\overline{X}^{\mathrm{tor}}_\Sigma}(\log\Delta)\right)$$

of the cotangent bundle of $\overline{X}^{\mathrm{tor}}_\Sigma$ resp. its logarithmic counterpart with irreducible boundary divisors in $\Delta = \overline{X}^{\mathrm{tor}}_\Sigma \setminus X(\Gamma)$ and is hard to compute.

Tsushima is able to compute this in the Siegel case by choosing the Γ-admissible family in such a way that the resulting toroidal compactification becomes the well-understood second Voronoi compactification. This is an exceptional case: It is a coincidence of low degree that this compactification is smooth and agrees with another well-known desingularization of the underlying Siegel modular variety; it does not generalize to other orthogonal modular varieties or higher degree.

Fiori's refinement

Fiori in his thesis and later work provides a more detailed description of the error term. The results of [Fio17] (in greater generality) yield tools reduce the error term to a linear combination of certain Euler characteristics on suitable subschemes of $\overline{X}_{\Sigma}^{\mathrm{tor}}$ and of intersection products involving divisors in $\overline{X}_{\Sigma}^{\mathrm{tor}} \setminus X(\Gamma)$ with high self-intersection. Moreover, it presents a way of handling these self-intersections, given the existence of a supply of suitable relations in the Chow ring of $\overline{X}_{\Sigma}^{\mathrm{tor}}$.

A closer characterization of these boundary divisors (or rather its open parts) of $\overline{X}_{\Sigma}^{\mathrm{tor}}$ is given in [Fio13]: These are quotients of toric varieties and fiber powers of universal elliptic curves. The exact geometry of the divisors and, with it, their intersection theory of the corresponding divisors depends heavily on the choice of the toroidal compactification $\overline{X}_{\Sigma}^{\mathrm{tor}}$, so any further computation of the error term would require an explicit and extensive knowledge of the geometry and combinatorics of a Γ-admissible family.

The lack of this is the central obstacle in Fiori's path to dimension formulas and its temporary endpoint: The classical existence theorems for admissible families by the theory of co-cores (cf. [AMRT10, Chapter 2]) are barely constructive and very inaccessible to a computational description.

This thesis will address these problems by describing a natural construction of Γ-admissible families for the orthogonal modular varieties $X(\Gamma)$ corresponding to the even unimodular lattice $II_{2,10}$ of signature $(2, 10)$ and carrying out many of the remaining steps to a computation of the error term:

We will describe the resulting toroidal compactification and the geometry and combinatorics of its divisor. Furthermore, we will develop parts of its intersection theory involving the toroidal boundary divisors and a certain class of divisors arising by the embedding of orthogonal modular varieties of lower rank. Finally, we will adapt and apply the tools of [Fio17] to this particular choice of toroidal compactification to get a formulation of the dimension formula in very explicit terms.

Additionally, this thesis may serve as a unified survey of several partial results in the theory of computation of dimension formulas for automorphic forms on lattices of signature $(2, n)$.

Main results

We summarize the main results of this thesis:

Let $II_{2,10}$ be the unique even unimodular lattice of signature $(2, 10)$. This lattice is remarkable as it is closely related to the lattices $II_{1,9}$ and E_8 by the decompositions

$$II_{2,10} = II_{1,1} \oplus II_{1,9} = II_{1,1} \oplus II_{1,1} \oplus \mathrm{E}_8(-1);$$

the $II_{1,9}$-lattice is a very special Lorentzian lattice of certain interest in the theory of Kac-Moody algebras of hyperbolic type while the E_8-lattice is a positive definite lattice which realizes the densest sphere packing in dimension 8. The discriminant kernel $\widetilde{\mathrm{SO}}^+(II_{2,10}(N))$ of the rescaling $II_{2,10}(N)$ for $N \geq 1$ can be considered as a finite-index subgroup of $\mathrm{O}^+(II_{2,10}) = \mathrm{O}^+(II_{2,10}(N))$.

The main objects of interest of this work are the spaces

$$S_{10k}\left(\widetilde{\mathrm{SO}}^+(II_{2,10}(N))\right)$$

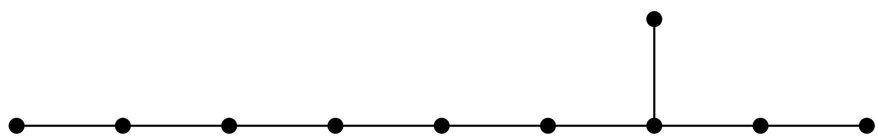
of orthogonal cusp forms of weight $10k$ on $\mathcal{D} = \mathcal{D}_{II_{2,10}} = \mathcal{D}_{II_{2,10}(N)}$ with respect to the group $\Gamma = \widetilde{\mathrm{SO}}^+(II_{2,10}(N))$ for suitable $N \geq 1$ and the corresponding orthogonal modular varieties

$$X(\Gamma) = \widetilde{\mathrm{SO}}^+(II_{2,10}(N))\backslash\mathcal{D}_{II_{2,10}}.$$

The reflective compactification

In view of the construction of toroidal compactifications efforts, the case of $L = II_{2,10}(N)$ has several advantages to the case of general lattices of signature $(2, n)$: The orthogonal group $\mathrm{O}^+(II_{2,10}(N))$ acts with a single orbit on the zero-dimensional rational boundary components of $\mathcal{D}$, so any $\Sigma(\mathcal{F})$ defines a Γ-admissible family by translation.

The following construction is central: Fix one of the zero-dimensional rational boundary components $\mathcal{F}$ and consider the cone $\mathcal{C}(\mathcal{F})$ and the group $\mathrm{Aut}(\mathcal{C}(\mathcal{F}))$. A result of Conway in [Con83] identifies the cone $\mathcal{C}(\mathcal{F})$ (up to a sign) as the positive timelike cone in the Lorentzian space $II_{1,9} \otimes \mathbb{R}$ and its group $\mathrm{Aut}(\mathcal{C}(\mathcal{F}))$ of autochronous symmetries as the reflection group $W(E_{10}(-1))$ with the Coxeter-Dynkin diagram



The hyperplanes corresponding to the reflections in $W(E_{10}(-1))$ decompose $\mathcal{C}(\mathcal{F})$ and its rational closure; by construction these decompositions are invariant under the action of any $\Gamma(\mathcal{F}) \subseteq \mathrm{Aut}(\mathcal{C}(\mathcal{F}))$ and the finiteness of the Coxeter-Dynkin diagram shows that there are only finitely many $\Gamma(\mathcal{F})$-orbits if $\Gamma(\mathcal{F}) \subseteq \mathrm{Aut}(\mathcal{C}(\mathcal{F}))$ is of finite index. This last property is a very special feature of the $II_{1,9}$-lattice. We call the resulting decomposition $\Sigma(\mathcal{F})$ the *Coxeter fan* and the induced family Σ the *Coxeter family.*

Our first main result is the following:

Theorem A (Theorem 11.2.9)**.** Let $L = II_{2,10}(N)$, $\Gamma = \widetilde{\mathrm{SO}}^+(II_{2,10}(N))$ its discriminant kernel and denote by $X = X(\Gamma) = \Gamma\backslash\mathcal{D}_L$ the corresponding orthogonal modular variety.

Let Σ be the family of cone decompositions induced by the Coxeter fan. This is a Γ-admissible family and defines a toroidal compactification $\overline{X}^{\mathrm{tor}}_{\Sigma}$, the *reflective compactification* of X. The reflective compactification $\overline{X}^{\mathrm{tor}}_{\Sigma}$ is projective for every $N \geq 1$ and has canonical singularities.

If $N = p > 13$ is prime, the reflective compactification is smooth.

This toroidal compactification is the basis of our computation of the dimension formula. The geometry and combinatorics of the reflective compactification is completely governed by the well-understood theory of the hyperbolic reflection group $W(E_{10})$.

Boundary and special divisors on the reflective compactification

We give a description of these *toroidal boundary divisors* as well as of the toroidal analogues of special cases of certain interior divisors: the *special divisors*. These arise as the embeddings of lower-dimensional orthogonal modular varieties and appear as the irreducible components of the divisors of Borcherds products.

As stated before, toroidal boundary divisors on orthogonal modular varieties come in two types. These are indexed by the boundary components, the *cusps*, of their Baily-Borel compactification, which are either points or modular curves and arise as Γ-classes of the projections of the rational boundary components of $\mathcal{D}$. Depending on the dimension of the associated boundary component, the toroidal boundary divisors are said to be of *zero-* resp. *one-dimensional type*.

We can describe toroidal boundary divisors and certain special divisors of the reflective compactification as follows:

Theorem B (Theorem 11.3.10)**.** Let $\overline{X}^{\mathrm{tor}}_{\Sigma}$ be the reflective compactification of a smooth $X\left(\widetilde{\mathrm{SO}}^+(I\!I_{2,10}(N))\right)$. Then:

- A toroidal boundary divisor of one-dimensional type over F is a smooth toroidal compactification of the Kuga-Sato variety $\mathcal{E}^{(n-2)}$ of rank $n-2$ over the modular curve F and the compactification (locally at a cusp F' of F) is defined by the star of the isotropic ray defining F inside the Coxeter fan associated to F'.
- A toroidal boundary divisor D of zero-dimensional type over F is isomorphic to a smooth compact toric variety corresponding to the star of a non-isotropic ray inside the Coxeter fan associated to F.
- Closures of certain special divisors (*reflective divisors*) in toroidal compactifications are toroidal compactifications themselves:

  Any $\lambda \in L$ with $q(\lambda) = -2N$ defines an orthogonal modular variety

  $$X(I\!I_{2,10}(N)\cap\lambda^{\perp}) = \widetilde{\mathrm{SO}}^+\left(I\!I_{2,10}(N)\cap\lambda^{\perp}\right)\backslash\mathcal{D}_{I\!I_{2,10}(N)\cap\lambda^{\perp}}.$$

  The natural morphism

  $$\widetilde{\mathrm{SO}}^+\left(I\!I_{2,10}(N)\cap\lambda^{\perp}\right)\backslash\mathcal{D}_{I\!I_{2,10}(N)\cap\lambda^{\perp}} \to \widetilde{\mathrm{SO}}^+\left(I\!I_{2,10}(N)\right)\backslash\mathcal{D}_{I\!I_{2,10}(N)},$$

  induced by the embedding $\mathcal{D}_{I\!I_{2,10}(N)\cap\lambda^{\perp}} \hookrightarrow \mathcal{D}_{I\!I_{2,10}(N)}$ of orthogonal symmetric domains, is a closed immersion and the closure of its image is isomorphic to the smooth projective toroidal compactification defined by the $\widetilde{\mathrm{SO}}^+(I\!I_{2,10}(N)\cap\lambda^{\perp})$-admissible family Σ_λ of cone decompositions obtained by restriction of Σ to $\lambda^{\perp}$:

  $$\overline{X(I\!I_{2,10}(N)\cap\lambda^{\perp})}^{\mathrm{tor}}_{\Sigma_\lambda} \cong \overline{X(I\!I_{2,10}(N)\cap\lambda^{\perp})} \subseteq \overline{X}^{\mathrm{tor}}_{\Sigma}$$

The last statement is based mainly on the work of Lan in [Lan19] and applies inductively to special cycles of higher codimension.

While the former two characterizations remain essentially true for general toroidal compactifications of orthogonal modular varieties corresponding to lattices of signature $(2, n)$, the latter result is a very special feature of the reflective compactification.

Intersection theory on the reflective compactification

The preceding description of the divisors on the reflective compactification is explicit enough to understand the corresponding part of the intersection theory of $\overline{X}_{\Sigma}^{\mathrm{tor}}$:

Theorem C (Theorem 12.2.15)**.** Let $\overline{X}_{\Sigma}^{\mathrm{tor}}$ be the reflective compactification of a smooth $X\left(\widetilde{\mathrm{SO}}^{+}(II_{2,10}(N))\right)$ for a suitable $N \geq 1$. The intersection theory of non-equal reflective divisors and boundary divisors can be described as follows:

- An intersection product of toroidal boundary divisors with each other is trivial if the associated cusps are not adjacent.
- An intersection of a toroidal boundary divisor with a reflective divisor is trivial if the associated cusp is not a cusp of the reflective divisor.

The remaining cases are described in the following intersection matrix, where D denotes the divisor in the respective column:

$\cdot$	divisor of one-dimensional type	divisor of zero-dimensional type	reflective special divisor
divisor of one-dimensional type	0	smooth compact toric variety inside D	divisor of one-dimensional type of D
divisor of zero-dimensional type	$*$	smooth compact toric variety inside D	divisor of zero-dimensional type of D
reflective special divisor	$*$	$*$	linear combination of reflective special divisors of D_1

Furthermore, we construct Borcherds products on $\overline{X}_{\Sigma}^{\mathrm{tor}}$ whose divisors consist of the components as described in the last theorem. These yield relations in the Chow ring that allow to treat intersection products with high self-intersection.

This is the knowledge needed for a successful application of the tools developed by Fiori and enables another take on the computation of the error term in the dimension formula.

Dimension formulas for $\Gamma = \widetilde{\mathrm{SO}}^{+}(II_{2,10}(N))$

An adapted version of Fiori's approach in [Fio17] combined with results of Tsushima in [Tsu80] allows to describe the error term more closely: Its linear coefficient depends solely on certain logarithmic Euler characteristics on pure self-intersection products of compactified Kuga-Sato varieties. We can recursively reduce these to expressions depending on certain geometric invariants of $X(\Gamma)$ and the modular curve $\Gamma(N)\backslash\mathbb{H}$. The constant coefficient depends similarly on these invariants, certain intersection products of toric varieties and the combinatorics of the reflection group $W(\mathrm{E}_{10})$.

Unfortunately, some of these expressions are very complicated and their computations requires a deeper knowledge of the exact structure of the Chow ring of $\overline{X}_{\Sigma}^{\mathrm{tor}}$ than we can provide at the moment, so further research is needed to arrive at a completely explicit error term.

Our final description of the dimension formula for orthogonal cusp forms on the lattice $II_{2,10}(N)$ is as follows:

Theorem D (Theorem 15.1.4)**.** Let $k \geq 2$ and $\Gamma = \widetilde{\mathrm{SO}}^{+}(II_{2,10}(N))$ neat.. The dimension of the space $\mathcal{S}_{10k}(\Gamma)$ of orthogonal cusp forms of arithmetic weight $10k$ on $X(II_{2,10}(N))$ is

$$\frac{[\mathrm{PO}^{+}(II_{2,10}) : \mathrm{P}\Gamma]}{92704053657600}\left[\binom{11k}{11k-11} - \binom{11k-2}{11k-13}\right] - c_1 \frac{\nu_1(2-2g+\nu_\infty)}{256}k + c_0$$

with $c_0, c_1 \in \mathbb{Q}$ certain rational numbers, ν_1 the number of one-dimensional Baily-Borel cusps of $X(II_{2,10}(N))$, g the genus of the modular curve $\Gamma(N)\backslash\mathbb{H}$ and ν_∞ the number of its cusps.

If $N = p \equiv 3 \mod 4$ is prime, the index and the numbers ν_1, ν_∞ and g can be computed explicitly.

The constants c_1 and c_0 depend on the geometry of the Coxeter family and certain recursive processes. These are described in chapter 13 and chapter 14, where methods and simplifications for their computations are given.

Even though this is by no means a full solution to the problem of dimension formulas for orthogonal modular forms on $\mathrm{O}(II_{2,10})$, it represents a further step towards a complete treatment.

Outline

We give a short overview of the structure of this thesis; it is divided into three parts which run roughly parallel on different levels of generality:

Part I treats the general theory of locally symmetric spaces, toroidal compactifications, automorphic forms and the problem of finding dimension formulas:

Chapter 1 introduces the notion of (locally) symmetric spaces an describes their geometry; it relates these varieties to the general framework of Shimura varieties and defines the concept of general automorphic forms.

Chapter 2 is a short introduction to toric geometry as this will be main tool in the construction of compactification of locally symmetric spaces.

Chapter 3 describes the construction of toroidal compactifications of locally symmetric spaces, lists many of its properties and gives some examples.

Chapter 4 deals with the general problem of finding dimension formulas for spaces of automorphic form. It introduces the central approach via the Hirzebruch-Riemann-Roch theorem and Hirzebruch-Mumford proportionality and describes certain properties of the non-canonical part of the dimension formula.

Part II moves the general theory to the domain of orthogonal locally symmetric spaces corresponding to even non-degenerate lattices of signature $(2, n)$ and explains the peculiarities of the general concepts from Part I in this situation:

Chapter 5 explains the general theory of even non-degenerate lattices, discriminant forms and orthogonal groups.

Chapter 6 is concerned with the special features of orthogonal (locally) symmetric spaces corresponding to lattices of signature $(2, n)$, their various models and geometry.

Chapter 7 is a closer treatment of the theory of orthogonal modular and cusp forms: It relates them to various other cases of automorphic forms and describes the Borcherds lift as a source of examples.

Chapter 8 describes the construction of toroidal compactifications of orthogonal locally symmetric spaces, the special case of symmetric compactifications and the possible singularities.

Chapter 9 examines the geometry of the most relevant divisors on orthogonal toroidal compactifications: the toroidal boundary divisors and special divisors.

Chapter 10 is the application of the approach in chapter 4 to the orthogonal setting: It computes the canonical part of the dimension formula and derives some properties of the error term.

Part III finally deals with the special choice of the $II_{2,10}$-lattice, constructs a special toroidal compactification and computes dimension formulas:

Chapter 11 reviews the theory of Coxeter and reflection groups, uses it to construct the reflective toroidal compactification of the orthogonal modular variety $X(II_{2,10}(N))$ and describes the divisors in this case.

Chapter 12 deals with the intersection theory of toroidal boundary divisor and special divisors on the reflective compactification.

Chapter 13 is concerned with the computation of the linear coefficient of the error term in the dimension formula.

Chapter 14 deals with the computation of the constant coefficients in the dimension formula and describes the necessary methods.

Chapter 15 finally collects all the preceding results and presents the results on the computation of the dimension formula, explains the remaining open problems and possible solutions.

Part I.

General theory

1. (Locally) Symmetric spaces and automorphic forms

We begin this thesis with a rather general survey of symmetric and locally symmetric spaces, their geometry, compactifications and automorphic forms on them, as well as their connection to the theory of Shimura varieties.
Locally symmetric spaces are interesting and rich objects. These spaces can be obtained as the quotient of a symmetric space by the action of an arithmetic subgroup, so they carry geometric, algebraic and arithmetic information.
We start with some very general definitions and facts and follow the treatment in [AMRT10] and [Hel01] for the symmetric and locally symmetric spaces. After this, we describe the geometry of these spaces and of a natural compactification. The second part of this chapter introduces the general theory of Shimura varieties and relates it to that of locally symmetric spaces. As a closing point, we introduce a first definition of automorphic forms in this generality.

1.1. (Locally) symmetric spaces and their geometry

The following is a rather short overview over the theory of symmetric and locally symmetric spaces. We begin with the notion of a symmetric space and assume the reader to be familiar with the basic concepts of Riemannian manifolds.
A symmetric space is a manifold modeled to have the property that its curvature tensor should be invariant under parallel transport. These spaces are well-studied objects in differential geometry, harmonic analysis and representation theory. The standard definition is the following:

Definition 1.1.1. A Riemannian manifold (M, g) is a *symmetric space* if for every point $p \in M$ there exists an *involution* σ_p at $p \in M$, i.e. an isometry with $\sigma_p(p) = p$ and $\mathrm{d}\sigma_p = -\operatorname{id}_{T_pM}$. If M is a Hermitian manifold, it is called a *Hermitian symmetric space*.

There is an equivalent description of symmetric spaces as quotients of isometry groups by maximal compacts:

Proposition 1.1.2 ([Hel01, Thm. IV.3.3])**.** *Let (M, g) be a symmetric space and let the isometry group $I(M)$ of M act transitively. Let K be the stabilizer of some point $p_0 \in M$. Then*

$$I(M)/K \cong M.$$

Conversely, let G be a connected Lie group and $K \subseteq G$ a closed subgroup with an involutive analytic automorphism σ of G such that

$$\mathrm{Fix}_\sigma(K)^0 \subseteq K \subseteq \mathrm{Fix}_\sigma(K)$$

and $\mathrm{Ad}_g(K)$ compact. Then G/K is a symmetric space.

Due to this result we are allowed to think of any symmetric space $\mathcal{D}$ as a quotient G/K of Lie groups as just stated.
Albeit the notion of general symmetric spaces is very rich, there is a structure theory that allows a coarse classification. Simply connected symmetric spaces can be decomposed into irreducible components, which in turn can be classified. An irreducible simply connected symmetric space $\mathcal{D}$ is of one of the following three types:

- *Euclidean type*: $\mathcal{D}$ has vanishing curvature (so it is isometric to a Euclidean space)
- *Compact type*: $\mathcal{D}$ has non-negative non-trivial curvature
- *Non-Compact type*: $\mathcal{D}$ has non-positive non-trivial curvature

Symmetric spaces of non-compact type are often called *symmetric domains* and turn out to be the objects of most interest to us.
We give a first example:

Example 1.1.3. A simple example of a Hermitian symmetric domain is given by the upper complex half-plane

$$\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

The automorphism group acts transitively on $\mathbb{H}$, so it suffices to give the involution at the single point $i \in \mathbb{H}$:

$$\tau \to \frac{-1}{\tau}.$$

As a quotient of Lie groups this is

$$\mathbb{H} \cong \mathrm{PSO}(2,1)(\mathbb{R})/\mathrm{SO}(2)(\mathbb{R}).$$

There is an associated symmetric space to any symmetric space, the so called *dual*, and there is a fully-developed duality theory for these spaces. The main point here is that the dual of a symmetric domain is of compact type and vice versa. For more results on these dual pairs the interested reader can consult [Hel01].
For our purposes, the following ad-hoc construction is sufficient:
Let $\mathcal{D} = G/K$ (via a base point p_0) be a symmetric domain and consider the Lie algebras $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{k} = \mathrm{Lie}(K)$. Denote the image of $\mathcal{D}$ under the canonical homomorphism $T_{p_0} \to \mathrm{Lie}(G)$ by $\mathfrak{p}$, then one has $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Set $\mathfrak{k}_c = \mathfrak{k}$, $\mathfrak{p}_c = i\mathfrak{p} \subset \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ and $\mathfrak{g}_c = \mathfrak{k}_c \oplus \mathfrak{p}_c$. The Lie groups corresponding to these Lie algebras are G_c, P_c resp. K_c and, as before, one gets a symmetric space $\check{\mathcal{D}} = G_c/K_c$ which is compact.

Definition 1.1.4. The symmetric space $\check{\mathcal{D}}$ is called the *compact dual of* $\mathcal{D}$.

In the following we will restrict ourselves even further to the case of Hermitian symmetric domains, so we assume any symmetric domain $\mathcal{D}$ to have the structure of a Hermitian manifold. The existence of such a complex structure on $\mathcal{D} \cong G/K$ can be read off the latter characterization as a quotient of Lie groups.

Proposition 1.1.5 ([BJ06, Proposition I.5.9])**.** *An irreducible symmetric space G/K is Hermitian if and only if K contains a central subgroup T isomorphic to the circle, i.e.*

$$T \cong \mathrm{SO}(2) \cong S^1.$$

One can use the decompositions of the Lie algebras of G and K to arrive at another realization of a Hermitian symmetric domain $\mathcal{D}$:
The complexification $\mathfrak{p}_{\mathbb{C}}$ of $\mathfrak{p}$ splits into a direct sum

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_{+} \oplus \mathfrak{p}_{-}$$

of ± 1-eigenspaces of the operator inducing the complex structures. For $X \in \mathfrak{p}_{+}$ let $T(X) : \mathfrak{p}_{-} \to \mathfrak{k}_{\mathbb{C}}$ be the map $Y \mapsto [X, Y]$ and $T^*(X) : \mathfrak{k}_{-} \to \mathfrak{p}_{\mathbb{C}}$ the adjoint with respect to a certain positive definite Hermitian form on $\mathfrak{g}_{\mathbb{C}}$, then:

Proposition 1.1.6 ([AMRT10, Chapter III, Theorem 2.9])**.** *A Hermitian symmetric domain $\mathcal{D}$ can be realized as*

$$\mathcal{D} \cong \left\{ X \in \mathfrak{p}_{+} \,|\, T^*(X) \circ T(X) < 2\,\mathrm{id}_{\mathfrak{p}_{-}} \right\}.$$

This realizes $\mathcal{D}$ as a bounded domain inside the complex vector space $\mathfrak{p}_{+}$. The corresponding embedding is called the *Harish-Chandra* embedding.
We turn our attention to the concept of *locally symmetric spaces.*

Locally symmetric spaces

While symmetric spaces are rich objects to study, our interest is of a more arithmetic nature, so we are interested in spaces that somehow incorporate arithmetic structure in their geometry, usually by quotienting out arithmetic symmetries.
The corresponding general notion is as follows:

Definition 1.1.7. A *locally symmetric space* is a Riemannian manifold (M, g) such that every point $p \in M$ has a neighborhood U with an isometry $\sigma_p : U \to U$ that is an involution at $p \in U$. It is called *Hermitian locally symmetric space* if the underlying Riemannian manifold is Hermitian.

The simplest example of this is given by the complex upper half-plane $\mathbb{H}$ and the quotient by the action of the integral special linear group (also called the *modular group*)

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : \, a, b, c, d \in \mathbb{Z} \text{ with } ad - bc = 1 \right\}$$

via
$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).\tau \mapsto \frac{a\tau + b}{c\tau + d}.$$
There is a result generalizing the construction of this example:

Proposition 1.1.8. *Let $\mathcal{D} = G/K$ be a symmetric space and let $\Gamma \subseteq G$ be a discrete subgroup that acts properly discontinuously. A space of the form $\Gamma\backslash G/K$ is a locally symmetric space. If Γ is torsion-free, this is smooth; in general, it has finite-quotient singularities.*

Conversely, one can show that any locally symmetric space is of the form $\Gamma\backslash G/K$ for a suitable Γ, cf. [BJ06].
These spaces are of arithmetic interest as they encode arithmetic information of the subgroup Γ into their geometry. As in the case of symmetric spaces, locally symmetric spaces come in a compact and a non-compact type, and many natural examples tend to be the latter, so a compactification theory is of great interest.

Baily-Borel compactifications

One would like to compactify locally symmetric spaces $\Gamma\backslash\mathcal{D}$ for a Hermitian symmetric domain $\mathcal{D}$ and an arithmetic $\Gamma \subseteq \mathrm{Aut}(\mathcal{D})$ (that is, discrete and acting properly discontinuous). A natural starting point would be a compactification of $\mathcal{D}$:
As seen in proposition 1.1.6 a Hermitian symmetric domain $\mathcal{D}$ can be realized as a bounded symmetric domain in $\mathfrak{p}_+$, so it makes sense to consider the closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ inside $\mathfrak{p}_+$. This amounts to the same objects as taking the closure of $\mathcal{D}$ inside the compact dual $\breve{\mathcal{D}}$.
The natural candidate for a compactification of $\Gamma\backslash\mathcal{D}$ is the space $\Gamma\backslash\overline{D}$. It is compact but exhibits a number of pathological behaviors.
This is due to the fact that the closure in $\mathfrak{p}_+$ adds 'too many' points to $\mathcal{D}$; this is to be expected as there is no relation between the boundary of $\overline{\mathcal{D}}$ (which is an analytic construction) and the quotient by Γ which is inherently arithmetic.
This can be remedied as follows: We assume that the Lie group $G = I(\mathcal{D})$ carries a lot more structure than just the analytic one making it a Lie group.
For the sake of completeness we recall the following standard notions in the theory of algebraic groups over the field $\mathbb{C}$ of complex numbers.

Definition 1.1.9. An algebraic group is a group which is an algebraic variety such that multiplication and inversion are regular maps on the variety.
The *radical* $R(G)$ of an algebraic group G is the identity component of its maximal solvable subgroup. Its subgroup $R_u(G)$ of *unipotent* elements (i.e. those $u \in R(G)$ with $(1-u)^n = 0$ for $n \gg 0$) is called the *unipotent radical*. An algebraic group is *semisimple* if its radical is trivial, and *reductive* if its unipotent radical is trivial. An algebraic group is called *linear* if its group of complex point is isomorphic to a subgroup of the group $\mathrm{GL}_n(\mathbb{C})$ for some $n \in \mathbb{N}$.

A *parabolic subgroup* $P \subseteq G$ is a Zariski-closed subgroup with the property that the quotient space G/P is a projective algebraic variety. A *real* parabolic subgroup is the group of real points of a parabolic subgroup. It is called *maximal* if it is maximal with respect to inclusion of subgroups with these properties. It is called *rational* if it is defined over $\mathbb{Q}$.

We assume from now on that the symmetric space $\mathcal{D} = G/K$ arises from a Lie group G that can be considered as the group $G(\mathbb{R})$ of real points of a semisimple linear algebraic group G defined over $\mathbb{Q}$.
Even though this seems to be a rather strong condition, almost all of the arithmetical examples of symmetric spaces fall in this category.
We need one further notion in the theory of algebraic groups:

Definition 1.1.10. An element $g \in \mathrm{GL}_n(\mathbb{Q})$ is called *neat*, if the subgroup of $\overline{\mathbb{Q}}^*$ generated by the eigenvalues of g is torsion-free. If G is a linear algebraic group defined over $\mathbb{Q}$, then an element $g \in G(\mathbb{Q})$ is called *neat* if its image in some faithful representation of G is neat. A subgroup of $G(\mathbb{Q})$ is called *neat* if all of its elements are neat.

In particular, a neat group is torsion-free and every subgroup of a neat subgroup is neat. The main reason to restrict oneself to these class of groups is that quotienting by the action of non-neat subgroups usually introduces finite-quotient singularities into the quotient which, in some circumstances, is to be avoided.
Our first example $\mathbb{H}$ with automorphism group $\mathrm{PSO}(2,1)$ certainly is in this category since the special orthogonal group is a linear algebraic group.
With this additional structure we can decompose the boundary of $\overline{\mathcal{D}}$ into irreducible components and then use the realization as a quotient of Lie groups to apply algebro-group-theoretic methods. This will give us a good characterization of the correct boundary components to add.
The following is the natural notion of *components* on $\overline{\mathcal{D}}$.

Definition 1.1.11. Two points $x, y \in \overline{\mathcal{D}}$ are said to be in the same *boundary component* if there exists a holomorphic map

$$\lambda : \mathbb{H} \to \overline{\mathcal{D}}$$

with $x, y \in \mathrm{Im}(\lambda)$. The boundary components of $\overline{\mathcal{D}}$ are the equivalence classes of this equivalence relation.

There is a one-to-one correspondence between boundary components of $\overline{\mathcal{D}}$ and maximal real parabolic subgroups of G.

Proposition 1.1.12 ([AMRT10, Proposition III.3.9])**.** *There is a bijective correspondence between boundary components of $\overline{\mathcal{D}} = \overline{G/K}$ and maximal real parabolic subgroups of G, realized by the map*

$$\mathcal{F} \mapsto \mathcal{P}(\mathcal{F})(\mathbb{R}) = \{g \in G(\mathbb{R}) \mid g\mathcal{F} = \mathcal{F}\} = \mathrm{Stab}_{G(\mathbb{R})}(\mathcal{F}).$$

This allows us to distinguish arithmetic boundary components from merely analytical boundary components. We call a boundary component of $\overline{\mathcal{D}}$ *rational* if its corresponding maximal parabolic subgroup is rational. These are the boundary components that yield an arithmetically meaningful compactification of $\Gamma\backslash\mathcal{D}$:

Theorem 1.1.13. *There exists a unique topology on*

$$\mathcal{D}^* = \mathcal{D} \sqcup \bigsqcup_{\mathcal{F} \text{ rational}} \mathcal{F}$$

with the property that the quotient

$$\overline{\Gamma\backslash\mathcal{D}}^{BB} = \Gamma \setminus \mathcal{D}^*$$

is a compactification of $\Gamma\backslash\mathcal{D}$ *and has the structure of a normal analytic space. Moreover, it is isomorphic to*

$$\mathrm{Proj}(M(\Gamma, \mathcal{D}))$$

where $M(\Gamma, \mathcal{D})$ *denotes a certain ring of functions on* $\mathcal{D}$*. This object is called the* Baily-Borel compactification *or* minimal Satake compactification*.*
It comes with the natural stratification by the equivalence classes of boundary components of $\mathcal{D}^*$*. We call* $\mathcal{D}$ *the trivial boundary component.*

The description of the Baily-Borel compactification via the Proj-construction shows that it is a projective variety over $\mathbb{C}$. As an open subset of $\overline{\Gamma\backslash\mathcal{D}}^{\mathrm{BB}}$ the quotient $\Gamma\backslash\mathcal{D}$ is a variety over $\mathbb{C}$ as well.

Remark 1.1.14. The ring of functions $M(\Gamma, \mathcal{D})$ is actually the ring of *modular forms* with respect to Γ on $\mathcal{D}$, a notion we have not introduced at this point. It will turn out to coincide with the notion of *automorphic forms* we are about to introduce in the last section of this chapter.

Example 1.1.15. In the case of $\mathcal{D} = \mathbb{H}$ this amounts to

$$\mathcal{D}^* = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}).$$

The corresponding quotient by $\mathrm{SL}_2(\mathbb{Z})$ adds a unique cusp usually denoted by $i\infty$ and yields a smooth compact complex curve.

When talking about locally symmetric spaces, we denote its boundary components by $F = \Gamma\mathcal{F}$ and use the notation $O(F)$ for any object $O(\mathcal{F})$ if this does not depend on the chosen representative of F.
While this is a reasonable compactification of $\Gamma\backslash\mathcal{D}$ for most purposes, it is often very singular. This poses problems for the application of Riemann-Roch type theorems. We will construct better compactifications by using more intricate properties of the geometry of $\Gamma\backslash\mathcal{D}$ to yield compactification without additional serious singularities.

Structure of parabolics

The structures of the parabolic subgroups will be of vital importance for our task of constructing better compactifications of locally symmetric spaces later on, so we will give a short summary of the structure theory of parabolic subgroups. For easier readability we will suppress the reference to the real points of the algebraic groups, so $G = G(\mathbb{R})$ and so on.

Theorem 1.1.16 ([AMRT10, Chapter III, Section 4]). *Let $\mathcal{P}$ be a maximal parabolic subgroup of a connected non-compact semisimple Lie group. Then $\mathcal{P}$ can be decomposed as*

$$\mathcal{P} = (\mathcal{U}(\mathcal{P}) \rtimes \mathcal{V}(\mathcal{P})) \rtimes (M(\mathcal{P}) \cdot G_h(\mathcal{P}) \cdot G_l(\mathcal{P}))$$

(the refined Langlands decomposition*) with $\mathcal{U}(\mathcal{P})$ the center of the unipotent radical $\mathcal{W}(\mathcal{P})$ of P and $\mathcal{V}(\mathcal{P}) = \mathcal{W}(\mathcal{P})/\mathcal{U}(\mathcal{P})$. The group $G_h(P)$ is semisimple, and, for $\mathcal{P} = \mathcal{P}(\mathcal{F})$ for some boundary component, its quotient by its center is equal to a connected component of* $\mathrm{Aut}(\mathcal{F})$*; the group $M(\mathcal{P})$ is compact while $G_l(\mathcal{P})$ is reductive without compact factors. Moreover, the group $G_l(\mathcal{P})$ is the automorphism group of a certain self-adjoint cone in $\mathcal{U}(\mathcal{P})$.*

If $\mathcal{P} = \mathcal{P}(\mathcal{F})$ (as will always be the case later on), we will write $O(\mathcal{F})$ for any object $O(\mathcal{P})$ as above.
The notation $G \cdot G'$ means the direct product modulo finite intersections. If any of the appearing groups is torsion-free, obviously this simplifies to the direct product.
The center $\mathcal{U}(\mathcal{P})$ of the unipotent radical $\mathcal{W}(\mathcal{P})$ of a real parabolic subgroup P as before is isomorphic to its Lie algebra $\mathfrak{u}$ and hence to a finite dimensional real vector space. The quotient $\mathcal{W}(\mathcal{P})/\mathcal{U}(\mathcal{P})$ of the unipotent radical by its center is also isomorphic to a finite-dimensional real vector space and furthermore can be equipped with the structure of a complex vector space.
One can use the rational boundary components and the corresponding maximal real parabolic subgroups with its decomposition to get another realization of Hermitian symmetric domains, the so-called *realization as a Siegel domain of the third kind.*
For the sake of completeness, we introduce the general notion of Siegel domains:

Definition 1.1.17. Let C be an open convex subset of $\mathbb{R}^m$ with $\mathbb{R}_+C = C$ (a *cone*).

1. A *Siegel domain of the first kind* is an open subset U of $\mathbb{C}^m$ of the form

$$U = \{z \in \mathbb{C}^m | \operatorname{Im}(z) \in C\}.$$

   This is often called a *tube domain.*

2. A *Siegel domain of the second kind* is an open subset $U \subseteq \mathbb{C}^n \times \mathbb{C}^m$ with

$$U = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m | \operatorname{Im}(z) - F(w, w) \in C\}$$

   for some Hermitian form $F : \mathbb{C}^n \times \mathbb{C}^n \to C$.

3. A *Siegel domain of the third kind* is an open subset $U \subseteq \mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C}^k$ with
$$U = \left\{(z,w,t) \in \mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C}^k |\ \mathrm{Im}(z) - F_t(w,w) \in C\right\}$$
for some semi-Hermitian form $F_t : \mathbb{C}^n \times \mathbb{C}^n \to C$ depending real-analytically on t, which itself comes from some bounded set in $\mathbb{C}^k$.

Obviously the higher types include the lower ones by setting $k = 0$ and $n = 0$. The simplest example is again given by $\mathbb{H}$ which is obviously a Siegel domain of the first kind with $C = \mathbb{R}_{>0}$.
One can show that Hermitian symmetric domains can be expressed exactly in this form by utilizing the decomposition of the maximal real parabolics. We recall the situation to fix notations, cf. [AMRT10, Chapter III, Section 4].
Let $\mathcal{F}$ be a rational boundary component of the Hermitian symmetric domain $\mathcal{D} = G/K$ and consider the decomposition
$$\mathcal{P}(\mathcal{F}) = (\mathcal{U}(\mathcal{F}) \rtimes \mathcal{V}(\mathcal{F})) \rtimes (M(\mathcal{F}) \cdot G_h(\mathcal{F}) \cdot G_l(\mathcal{F}))\,.$$

Proposition 1.1.18. *There is an open homogeneous cone $\mathcal{C}(\mathcal{F}) \subseteq \mathcal{U}(\mathcal{F})$ which is self-adjoint with respect to the positive-definite natural quadratic form on $\mathfrak{u}(\mathcal{F})$. We have*
$$\mathcal{C}(\mathcal{F}) \cong G_l(\mathcal{F})/G_l(\mathcal{F}) \cap K$$
and
$$\mathcal{F} \cong G_h(\mathcal{F})/G_h(\mathcal{F}) \cap K.$$

There is a natural embedding of $\mathcal{D}$ into its compact dual $\breve{\mathcal{D}}$ and one can consider the action of $\mathcal{U}(\mathcal{F})_{\mathbb{C}} := \mathcal{U}(\mathcal{F}) \otimes \mathbb{C}$ on $\mathcal{D} \subseteq \breve{\mathcal{D}}$ by the natural action on the Lie algebras and define
$$\mathcal{D}(\mathcal{F}) = \mathcal{U}(\mathcal{F})_{\mathbb{C}} \cdot \mathcal{D} \subseteq \breve{\mathcal{D}}.$$

Proposition 1.1.19. *In the preceding situation we have an isomorphism*
$$\mathcal{D}(\mathcal{F}) \cong \mathcal{U}(\mathcal{F})_{\mathbb{C}} \times \mathcal{V}(\mathcal{F}) \times \mathcal{F}$$
and the inclusion $\mathcal{D} \subseteq \mathcal{D}(\mathcal{F})$ is via
$$\mathcal{D} \cong \{(x,y,z) \in \mathcal{U}(\mathcal{F})_{\mathbb{C}} \times \mathcal{V}(\mathcal{F}) \times \mathcal{F} |\ \mathrm{Im}\, x - h_z(y,y) \in \mathcal{C}(\mathcal{F})\}$$
for a certain real bilinear quadratic form h_z on $\mathcal{V}(\mathcal{F}) \cong \mathbb{C}^k$ for some $k \in \mathbb{Z}$, depending real-analytically on $z \in \mathcal{F}$.

In particular, any Hermitian symmetric domain is a Siegel domain of the third kind.
One can show that the rational boundary components $\mathcal{F}' \subset \overline{\mathcal{F}}$ correspond bijectively to rational isotropic cones $\mathcal{C}(\mathcal{F}')$ in the boundary of $\mathcal{C}(\mathcal{F})$.
The realizations as Siegel domains of the third kind will be of vital importance for the compactification of locally symmetric spaces.
This ends our first treatment of locally symmetric spaces and their geometry. We turn to its relation to the important theory of Shimura varieties.

1.2. Shimura varieties and automorphic forms

Locally symmetric varieties are closely related to the ubiquitous concept of Shimura varieties, one of the central notions of modern number theory, and to automorphic forms, another important class of objects. We give brief introductions to the general concepts, deepening the treatment in later chapters whenever needed. The treatment here follows the one in [Lan16].

Shimura varieties

We give a very short and incomplete sketch of the notion of Shimura varieties. As some of the literature needed later on is written in the language of Shimura data, we deem it useful to introduce the most common concepts.
Let G be a reductive algebraic group over $\mathbb{Q}$ and $\mathcal{D}$ a manifold with smooth transitive $G(\mathbb{R})$-action. Let $\mathcal{U}$ be any neat open compact subgroup of $G(\mathbb{A}_f)$, with $\mathbb{A}_f$ denoting the finite adeles. Define the double coset space

$$X_{\mathcal{U}} := G(\mathbb{Q})\backslash(\mathcal{D} \times G\left(\mathbb{A}_f\right))/\mathcal{U}.$$

We note that

$$G(\mathbb{A}_f) = \coprod_{i\in I} G(\mathbb{Q})^+ g_i \mathcal{U}$$

for some finite set I. We set

$$\Gamma_i := \left(g_i \mathcal{U} g_i^{-1}\right) \cap G(\mathbb{Q})^+,$$

and denote by $\mathcal{D}^+$ a connected component of $\mathcal{D}$ with transitive $G(\mathbb{R})$-action, then

$$X_{\mathcal{U}} = \coprod_{i\in I} \Gamma_i\backslash\mathcal{D}$$

with all Γ_i being arithmetic subgroups of $G(\mathbb{Q})$, so $\Gamma_i\backslash\mathcal{D}^+$ is a locally symmetric space in the sense of the last section. Quotients arising via this kind of construction are called *connected Shimura varieties*.
In general, Shimura varieties arise from so-called *Shimura data*. A Shimura datum is a pair $(G, \mathcal{D})$ with G a connected reductive algebraic group and $\mathcal{D}$ is a $G(\mathbb{R})$-conjugacy class of homomorphisms

$$\mathbb{S} \to G_{\mathbb{R}}$$

from the *Deligne torus* $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ (whose group of real points is isomorphic to $\mathrm{U}_1 \cong \mathbb{C}^*$ and the group of complex points to $\mathbb{C}^* \times \mathbb{C}^*$) to the base change $G_{\mathbb{R}}$ of G to $\mathbb{R}$, satisfying three additional conditions. We will not go into further details about this construction here.
These conditions force $\mathcal{D}$ to be a finite union of Hermitian symmetric domains and any choice of an open neat compact $\mathcal{U}$ as before yields locally symmetric spaces $\Gamma_i\backslash\mathcal{D}^+$ which are also smooth varieties.

The main point of the additional conditions is the fact that these force the quotients $X_{\mathcal{U}}$ for any neat open compact $\mathcal{U}$ to have a *canonical model* as a variety over a number field. This is important for the general theory but not for our purposes. For clarification: The connected components $\Gamma_i \backslash \mathcal{D}$ of $X_{\mathcal{U}}$ are sometimes called *Shimura varieties* at level $\mathcal{U}$ to emphasize their dependence on the choice of $\mathcal{U}$.
Morphisms between Shimura data $(G_1, \mathcal{D}_1), (G_2, \mathcal{D}_2)$ are simply group homomorphism $G_1 \to G_2$ sending $\mathcal{D}_1$ into $\mathcal{D}_2$.
When $G_1 \to G_2$ is injective and $\mathcal{U}_1$ is the pullback of $\mathcal{U}_2$ via $G_1(\mathbb{A}^\infty) \to G_2(\mathbb{A}^\infty)$, the Shimura variety $X_{\mathcal{U}_1}$ is a closed subvariety of $X_{\mathcal{U}_2}$ and is called a *special subvariety*.
There is a classification of Shimura data which in some sense measure the tractability of handling the corresponding Shimura varieties. The simplest case is that of Shimura data modeled after the *general symplectic group* GSp_{2n} and the *Siegel half-plane* $\mathcal{H}_n$. These are said to be of *PEL type*, since they describe moduli problems of abelian varieties with a *P*olarization, an *E*ndomorphism structure and a *L*evel.
A more general type of Shimura data $(G, \mathcal{D})$ is given by those of *Hodge type*, these are characterized by having an injective homomorphism $G \hookrightarrow \mathrm{GSp}_{2n}$ inducing an embedding $\mathcal{D} \hookrightarrow \mathcal{H}_n$ for some $n \geq 0$.
Note that there is also the closely related notion of connected Shimura data which yields $|I| = 1$ in the preceding consideration. Moreover, the Shimura varieties coming from a connected Shimura datum $(G, \mathcal{D})$ depend only on the adjoint group $G^{\mathrm{ad}} = G/Z(G)$ of G, so central extensions of G yield the same Shimura varieties.
We will employ the language developed here later on and use it interchangeably with the language of locally symmetric spaces whenever we work with an locally symmetric space which is also a Shimura variety (which will actually always be the case).

Automorphic forms

Automorphic forms are closely related to locally symmetric spaces and hence to Shimura varieties.
The following is a preliminary characterization of the concept of *automorphic forms* in a very broad sense. This serves just to help intuition. We will give more specialized and precise definitions in a moment.
Let G be a semisimple real Lie group with finite center of non-compact type, K a maximal compact subgroup, Γ a discrete subgroup and V a finite-dimensional vector space with a representation $\rho : G \to \mathrm{GL}(V)$. An automorphic form on G is a function $f : G \to V$ such that

- $f(\gamma g k) = j_\gamma(g)\rho(k)^{-1} f(g)$ for $\gamma \in \Gamma, k \in K$ with j_γ the *factor of automorphy*
- f is an eigenfunction of the G-invariant differential operators on G/K
- f is of moderate growth

Note that any one of these conditions is ill-defined at this point.

The first of these is responsible for the name, since it shows self-similarity (Greek: ¨αὐτός' - self and ¨μορφή'- shape) of f under transformation. The second condition is a generalized form of analytic well-behavior, while the third is a technical condition to ensure extendability to compactification and finite-dimensionality of the resulting spaces.
On bounded symmetric domains $\mathcal{D} = G/K$, the following is the viable version of the preceding concept, cf. [Mum77, §3]:

Definition 1.2.1. A holomorphic ρ-automorphic form f of weight 1 for a representation $\rho : K \to \mathrm{GL}_n(\mathbb{C})$ is a holomorphic function $f : G \to \mathbb{C}^n$ with

$$f(\gamma g k) = \rho(k)^{-1} f(g),$$

satisfying

$$|f(g)| \leq \mathrm{tr}\left(\mathrm{Ad} g^{*-1}.g\right)^n$$

for $*$ the Cartan involution and some $n \gg 0$. Moreover, f induces a holomorphic section of the bundle

$$E_0 = G \times_K \mathbb{C}^n = G \times \mathbb{C}^n \Big/ \left[(g, x) \sim \left(kg, \rho(k)^{-1} x\right)\right]$$

over $\mathcal{D} = G/K$ (this bundle has a natural complex structure). A ρ-automorphic form of weight k is analogously defined by the same conditions except inducing a holomorphic section of $E_0^{\otimes k}$. If f vanishes on the boundary of $\mathcal{D}$ (inside the compact dual), it is called a ρ-automorphic cusp form.

We denote the vector space of the ρ-automorphic forms of weight k on by $\mathcal{M}_k^\rho(\Gamma)$ and the space of ρ-cusp forms by $\mathcal{S}_k^\rho(\Gamma)$. These forms are sometimes called *vector valued* if $n > 1$.
If $n = 1$ and hence $\rho = \mathrm{id}$ (the *scalar-valued* case) the definition simplifies: An automorphic form f is a holomorphic Γ-invariant form $f : \mathcal{D} = G/K \to \mathbb{C}$ on the symmetric space $\mathcal{D}$ with some natural growth condition. The corresponding spaces of automorphic forms and cusp forms will be denoted by $\mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k(\Gamma)$, respectively.
In this case, the bundle E_0 is simply the canonical line bundle, so automorphic forms can be considered as holomorphic sections of the descent of E_0 to $\Gamma\backslash\mathcal{D}$. This reformulates the concept of an automorphic form in geometric terms on the locally symmetric space $X = \Gamma\backslash\mathcal{D}$ (and therefore on Shimura varieties).
We note that this is a special case of the ill-defined general notion: Such an f transforms with trivial factor of automorphy and the Casimir element is a scalar multiple of the Laplacian of $\mathcal{D}$: Since f is holomorphic, it is an eigenfunction of this operator with eigenvalue 0.

Remark 1.2.2. The reformulation of automorphic forms as sections of bundles has been extremely influential, so Deligne's notion of an automorphic form even takes this as a definition: An automorphic form is a section of a bundle on a Shimura variety.

Such automorphic forms behave well with respect to the Baily-Borel compactification: An automorphic form on X extends to the Baily-Borel compactification $\overline{X}^{\mathrm{BB}}$ and cusp forms are exactly those automorphic forms vanishing on $\partial\overline{X}^{\mathrm{BB}} = \overline{X}^{\mathrm{BB}} \setminus X$.
This ends our first exposition of the theory of (locally) symmetric spaces and automorphic forms. The next chapter will introduce the theory of *toric geometry* which is the central tool for the construction of improved versions of the Baily-Borel compactifications.

2. Toric geometry

We noted in chapter 1 that the many natural locally symmetric spaces are non-compact and that the Baily-Borel compactification is highly singular in general. In [Mum73] Mumford devised a new approach for the construction of smooth compactifications of locally symmetric spaces. The main tool in this strategy is the use of toric varieties whose theory we will introduce here.
We start this section by recalling some of the basics of convex geometry, then moving on to the general theory of toric varieties and their properties. Good references about toric geometry (and where most of the statements here are taken of) are [CLS11] (for finite cone decompositions) or [Oda78] for the general case. The second part of this chapter deals with the notion of fiber products in toric geometry.

2.1. Basic toric geometry

The foundational concept in this theory is the notion of an *algebraic torus*. We give its full definition as an algebraic group.

Definition 2.1.1. Let F be a field. The *multiplicative group over F* is the algebraic group $\mathbb{G}_m$ such that $\mathbb{G}_m(E) \cong E^*$ for any field extension E/F.
Denote an algebraic closure of F by $\overline{F}$. An *F-torus* is an algebraic group T over F such that there is an $r \geq 0$ with

$$T\left(\overline{F}\right) \cong \left(\mathbb{G}_m\left(\overline{F}\right)\right)^r \cong \left(\overline{F}^*\right)^r.$$

This notion appeared briefly in the definition of Shimura data in section 1.2 via the Deligne torus. This is not the same as the notion of the topological tori $(S^1)^r$. A complex algebraic torus is just $(\mathbb{C}^*)^r$ for some $r \geq 0$.
These objects are very canonical and easy to understand, so it is natural to consider objects conceptually close to them. The notion of toric varieties hence arises as those varieties being almost a torus:

Definition 2.1.2. A *toric variety* is an irreducible variety X containing an algebraic torus T as a Zariski-dense subset such that the action

$$T \times T \to T$$

of T on itself extends to a morphism

$$T \times X \to X.$$

In the following we will work with complex toric varieties, so $\overline{F} = \mathbb{C}$ and an algebraic torus is isomorphic to $(\mathbb{C}^*)^r$ for some $r \geq 0$. Standard examples of toric varieties are affine and projective n-space via

$$(\mathbb{C}^*)^n \subset \mathbb{C}^n \subset \mathbb{P}_{\mathbb{C}}{}^n.$$

One of the most important features of these varieties is the fact that their geometric features can be equivalently described by combinatorial data. This data can be obtained as follows:
We denote by $X^*(T)$ the group of co-characters of T, that is, the set $\mathrm{Hom}(\mathbb{C}^*, T)$ of one-parameter subgroups in T. Furthermore we denote by $X_*(T)$ the group $\mathrm{Hom}(T, \mathbb{C}^*)$ of characters of T. Both $X^*(T)$ and $X_*(T)$ carry the structure as a free $\mathbb{Z}$-module of rank n. Since $\mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{Z}$ we get a non-degenerate bilinear pairing

$$\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$$

by composition and the above isomorphism:

$$(u, v) \mapsto u \circ v \in \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{Z}.$$

This bilinear form extends naturally to a non-degenerate bilinear form

$$\langle \cdot, \cdot \rangle : X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \times X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}.$$

The above construction associates to every complex algebraic torus T a lattice $X^*(T)$ and its dual $X_*(T)$. There is an equally natural construction for the reverse direction: For a lattice L of rank n we can form the quotient

$$T_L = (T \otimes \mathbb{C}) / T \cong \mathbb{C}^n / \mathbb{Z}^n \cong (\mathbb{C}^*)^n$$

by choosing a basis for L and applying the exponential function.
We use the following general notation: For any $\mathbb{Z}$-module M we will denote its tensor product $M \otimes_{\mathbb{Z}} \mathbb{R}$ with $\mathbb{R}$ by $M_{\mathbb{R}}$.
The real vector space $X^*(T)_{\mathbb{R}}$ is the natural domain for the combinatorial description. We introduce some language to describe the appearing combinatorial objects:

Definition 2.1.3. Let V be a finite-dimensional real vector space. A *cone* $\mathcal{C} \subseteq V$ is a subset satisfying

$$\mathbb{R}^+ \mathcal{C} \subseteq \mathcal{C}.$$

It is called *convex* if it is closed under addition, so $x, y \in \mathcal{C}$ implies $x + y \in \mathcal{C}$, and *strongly convex* if it is convex and does not contain a non-trivial linear subspace. It is called *polyhedral* if there exist $v_1, \ldots, v_n \in V$ with

$$\mathcal{C} = \mathbb{R}_{\geq 0} v_1 + \ldots + \mathbb{R}_{\geq 0} v_n.$$

If $V = L \otimes \mathbb{R}$ for some lattice L, the cone $\mathcal{C}$ is called *rational* if $v_1, \ldots, v_n$ can be chosen in L.

The polyhedrality condition implies that a polyhedral cone is closed. This is not necessary for the general theory and there are formulations working with open polyhedral cones

$$\mathbb{R}_{>0}v_1 + \ldots + \mathbb{R}_{>0}v_n.$$

The definition of convexity given above implies the more standard one

$$x, y \in X \Longrightarrow \lambda x + (1-\lambda)y \in X \quad \text{for every } \lambda \in (0,1).$$

To save writing effort, in the following a cone is assumed to be strongly convex and polyhedral.
Polyhedral cones carry the combinatorial structure of their face configuration with them:

Definition 2.1.4. A *face* of a cone $\mathcal{C}$ is the intersection of $\mathcal{C}$ with a *supporting hyperplane* $\mathcal{H}$, that is, a hyperplane such that $\mathcal{C} \cap \mathcal{H}^c$ is still connected.

The face of a cone is a cone itself. If σ is a face of $\mathcal{C}$, we write

$$\sigma \preceq \mathcal{C}.$$

One sees easily that the set of faces of $\mathcal{C}$ is closed under the operation of taking faces. It is clear that a polyhedral convex cone $\mathcal{C}$ has only finitely many faces and each of these is again a polyhedral cone. Any intersection of faces is also again a face.
There is also the notion of the dual cone:

Definition 2.1.5. For any cone $\sigma \in V$ the *dual cone* is

$$\sigma^\vee = \{v \in V^* | v(u) \geq 0 \text{ for all } u \in \sigma\}$$

The dual of a strongly convex rational polyhedral cone is again a strongly convex polyhedral cone by *Farkas' lemma*.
The following is the central construction of the combinatorial theory of toric varieties: For a strongly convex rational polyhedral cone $\sigma \subseteq X_*(T)_\mathbb{R}$ we define the variety

$$X_\sigma = \text{Spec } \mathbb{C}[X^*(T) \cap \sigma^\vee].$$

This is an algebraic variety by the fundamental *Gordan's lemma*:

Proposition 2.1.6 ([Ful98, Section 1.2, Proposition 1])**.** *Let $\sigma \subseteq X^*(T)$ be a strongly convex rational polyhedral cone. Then $\sigma^\vee \cap X^*(T)$ is a finitely generated semigroup in $X^*(T)$. In particular, the group ring $\mathbb{C}[\sigma^\vee \cap X^*(T)]$ is a $\mathbb{C}$-algebra.*

The variety X_σ is called an *affine toric variety.* It contains a Zariski-dense algebraic torus by the natural morphism

$$(\mathbb{C}^*)^{\text{rank}(X^*(T))} \cong \text{Spec}\,(\mathbb{C}[X^*(T)]) \to \text{Spec}\,(\mathbb{C}[X^*(T) \cap \sigma^\vee]) = X_\sigma$$

induced by the inclusion

$$\mathbb{C}[X^*(T) \cap \sigma^\vee] \subseteq \mathbb{C}[X^*(T)]$$

whose action on itself extends to X_σ.
As usual, a more complex notion arises by gluing affine objects together to get locally affine objects. The gluing data is organized in *rational polyhedral partial decompositions*:

Definition 2.1.7. A *rational polyhedral partial decomposition* Σ of a cone $\mathcal{C} \subseteq X_*(T)$ is a collection Σ of cones, satisfying the following conditions:

(i) Each cone $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.

(ii) Every face of every $\sigma \in \Sigma$ is again in Σ.

(iii) For $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

If the collection Σ is finite, this is a *finite* rational polyhedral partial cone decomposition.

These objects are often called *fans* in the literature. We will use this name interchangeably with *rational polyhedral partial decomposition.*
We can associate a variety to a rational polyhedral partial decomposition Σ in the following way: For τ a face of σ we have a monomorphism of semigroups

$$k[X^*(T) \cap \sigma^\vee] \hookrightarrow k[X^*(T) \cap \tau^\vee]$$

and hence an embedding $X_\tau \hookrightarrow X_\sigma$. Gluing the collection $\{X_\sigma | \sigma \in \Sigma\}$ via the open embeddings of their pairwise intersections (meaning we glue X_σ and X_τ on the open subset $X_{\sigma\cap\tau}$ which is a subset of both) gives a variety locally of finite type which we denote by X_Σ.
This variety contains the original torus T as $T \cong \mathrm{Spec}(X^*(T)) = X_{\{0\}}$, corresponding to the zero cone which is a face of every cone in Σ. It is a toric variety in the sense of definition 2.1.2. If the rational polyhedral partial decomposition is finite, the variety X_Σ is of finite type.
The preceding construction is universal in the following sense:

Proposition 2.1.8 ([Oda78, Theorem 4.1])**.** *Let Σ be a fan in $X^*(T)_\mathbb{R}$. The variety X_Σ is a normal separated toric variety. Any normal separated toric variety X locally of finite type arises as $X = X_\Sigma$ for a suitable fan Σ in $X^*(T)_\mathbb{R}$.*
The toric variety is of finite type if and only if Σ is finite.

Properties

One of the great strengths of toric geometry is the possibility to translate between geometric properties of toric varieties and combinatorial properties of the defining fans. An example is the following: The geometric notion of smoothness can be characterized via linear independence of the spanning rays of the cones.

Lemma 2.1.9 ([Oda78, Theorem 4.3])**.** *Let Σ be a fan in $X^*(T)_\mathbb{R}$. The toric variety X_Σ is smooth if and only if the fan Σ is smooth, that is, every cone $\sigma \in \Sigma$ is generated by part of a $\mathbb{Z}$-basis of $X^*(T)$.*

While this lemma shows that the local property of smoothness is governed by local properties of the fan, there are also results of a more global nature:

Lemma 2.1.10 ([Oda78, Corollary 4.5])**.** *Let Σ be a fan in $X^*(T)_{\mathbb{R}}$. The toric variety X_Σ is compact if and only if the fan Σ is finite and its support $|\Sigma| = \bigcup_{\sigma\in\Sigma}\sigma$ is $X^*(T)$.*

Going back to the very definition of a toric variety X, we remember that there is a natural action of the torus $T \subseteq X$ on X, so X is the disjoint union of the orbits of this action. Later on, we will identify certain objects as torus orbits of toric varieties, hence we'll collect the basic results on these for reference:

Proposition 2.1.11 ([Oda78, Theorem 4.2])**.** *Let X_Σ be the toric variety of the fan Σ in $X^*(T)_{\mathbb{R}}$. There is a bijective correspondence*

$$\begin{aligned} \{\textit{cones } \sigma \textit{ in } \Sigma\} \quad &\longleftrightarrow \quad \{T\textit{-orbits in } X_\Sigma\} \\ \sigma \quad &\longleftrightarrow \quad O(\sigma) = \mathrm{Hom}_{\mathbb{Z}}\left(\sigma^\perp \cap X_*(T), \mathbb{C}^*\right) \end{aligned}$$

and the closure of every torus orbit $O(\tau)$ is determined by the cones containing τ

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma \in \Sigma} O(\sigma).$$

There is an easy way to describe the closure of a given torus orbit in terms of the defining fan. To formulate the statement, we need to introduce the notion of the star of a cone $\tau \in \Sigma$ inside the fan Σ.

Definition 2.1.12. Let Σ be a fan in $V = L \otimes \mathbb{R}$ for a lattice L and let $\tau \in \Sigma$ a cone. Define L_τ as the $\mathbb{Z}$-span of $\tau \cap L$ and denote the quotient group L/L_τ by $L(\tau)$. For each cone $\sigma \in \Sigma$ let $\overline{\sigma}$ be the image of σ under the quotient map

$$V \to V(\tau) = L(\tau) \otimes \mathbb{R} \cong V/L_\tau \otimes \mathbb{R}$$

and define the *star* $\mathrm{Star}_\Sigma(\tau)$ *of* $\tau \in \Sigma$ to be

$$\mathrm{Star}_\Sigma(\tau) = \{\overline{\sigma} \subseteq V(\tau) |\ \tau \text{ is a face of } \sigma\}.$$

The star $\mathrm{Star}_\Sigma(\tau)$ is again a fan, this time in the quotient vector space $V(\tau)$. We will sometimes meddle the notation and use the notion of the star of a cone also for the set of cones containing it as face (instead of its image under the projection). The usage will always be clear from context as the the latter interpretation concerns cones in V while the former is about cones in its quotient $V(\tau)$.
With this notion we can describe the geometry in a toric variety quite closely:

Lemma 2.1.13. *For any $\tau \in \Sigma$, the orbit closure $V(\tau) = \overline{O(\tau)}$ in X_Σ is isomorphic to the toric variety $X_{\mathrm{Star}_\Sigma(\tau)}$ defined by the star $\mathrm{Star}_\Sigma(\tau)$ of τ in Σ.*

Proof. This is an immediate consequence of proposition 2.1.11, see also [CLS11, §3.2] or again [Oda78, Theorem 4.2] in the possibly infinite case. □

There is also a categorial interpretation of toric varieties. Having defined the objects of this category, we are still missing the morphisms:

Definition 2.1.14. Let $X_{\Sigma_1}, X_{\Sigma_2}$ be normal toric varieties with corresponding fans Σ_1, Σ_2 in $X^*(T_1)$ resp. $X^*(T_2)$. A morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is called *toric morphism* if $\phi(T_1) \subseteq T_2$ and $\phi|_{T_1}$ is a group homomorphism.

This is the natural notion of morphisms in the category of toric varieties.
As always in toric geometry, there is a completely algebraic characterization of toric morphisms in term of the associated fans. To formulate it, we need the notion of maps *compatible* with the given lattice structure.

Definition 2.1.15. Let $X^*(T_1), X^*(T_2)$ be lattices and Σ_1, Σ_2 fans in the corresponding real vector spaces $X^*(T_1)_{\mathbb{R}}, X^*(T_2)_{\mathbb{R}}$. We call a $\mathbb{Z}$-linear mapping $\phi : X^*(T_1) \to X^*(T_2)$ *compatible* (with the fans Σ_1, Σ_2) if for any $\sigma_1 \in \Sigma_1$ there exists $\sigma_2 \in \Sigma_2$ containing $\phi(\sigma_1)$.

These maps are the morphisms in the category of rational polyhedral partial decompositions.
We have the following theorem translating the algebraic definition of toric morphism into combinatorial data (and vice versa):

Proposition 2.1.16. *Let $X^*(T_1), X^*(T_2)$ be lattices and Σ_1, Σ_2 fans in the corresponding real vector spaces $X^*(T_1)_{\mathbb{R}}, X^*(T_2)_{\mathbb{R}}$.*

- *A compatible mapping $\phi : X^*(T_1) \to X^*(T_2)$ induces a canonical toric morphism $\psi : X_{\Sigma_1} \to X_{\Sigma_2}$ such that $\psi|_{T_1}$ is*

$$\phi \otimes 1 : X^*(T_1) \otimes_{\mathbb{Z}} \mathbb{C} \to X^*(T_2) \otimes_{\mathbb{Z}} \mathbb{C}.$$

- *A toric morphism $\psi : X_{\Sigma_1} \to X_{\Sigma_2}$ induces a compatible map $\phi : X^*(T_1) \to X^*(T_2)$.*

This upgrades the result of proposition 2.1.8 to an equivalence of categories between rational polyhedral partial decompositions and normal separated varieties locally of finite type. It allows us to transfer morphisms defined in terms of fans to morphisms between toric varieties.
We want to examine this in the simplest example of a single endomorphism of $X^*(T)_{\mathbb{R}}$ compatible with two different fans. To get a compatible morphism, one of the fans is required to *refine* the other:

Definition 2.1.17. Let Σ, Σ' be fans in $X^*(T)_{\mathbb{R}}$. The fan Σ' is a *refinement* of Σ if

- for every $\sigma' \in \Sigma'$ there exists a $\sigma \in \Sigma$ with $\sigma' \subseteq \sigma$
- both fans have the same *support*, i.e.

$$|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma = \bigcup_{\sigma' \in \Sigma'} \sigma' = |\Sigma'|.$$

This induces a pre-order on the set of fans with same support by

$$\Sigma_1 \preceq \Sigma_2 \text{ if and only if } \Sigma_2 \text{ is a refinement of } \Sigma_1.$$

Fortunately, the set of fan with given support is directed, that is, for any two fans Σ_1, Σ_2 with given support $|\Sigma_1| = |\Sigma_2|$ there exist fans Σ_3 with the same support, refining both of them. The minimal one of them is called the *coarsest common refinement.*

Definition 2.1.18. Let Σ_1, Σ_2 be fans with support $|\Sigma_1| = |\Sigma_2|$. Define

$$\Sigma_1 \wedge \Sigma_2 \colon = \{\sigma_1 \cap \sigma_2 | \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2\}.$$

This set is called the *coarsest common refinement* of Σ_1 and Σ_2.

This set is a fan and refines both Σ_1 and Σ_2. It is the coarsest refinement in the following sense: Any refinement with the same properties is a refinement of $\Sigma_1 \wedge \Sigma_2$.
Let Σ' be a refinement of the fan Σ as above. Then the identity map $\mathrm{id} : X^*(T) \to X^*(T)$ is compatible with Σ and Σ' and gives rise to a toric morphism $\psi : X_\Sigma \to X_{\Sigma'}$ by proposition 2.1.16. Since it is an isomorphism of the open torus T contained in both varieties, it is birational.
For later reference, we formulate this as a result:

Lemma 2.1.19. *Let Σ' be a refinement of a fan Σ. There is a natural birational morphism*

$$X_\Sigma \to X_{\Sigma'}.$$

By the following result, one can say even more about this morphism in certain cases:

Lemma 2.1.20. *Let $\psi : X^*(T_1) \to X^*(T_2)$ be compatible for the fans Σ_1 and Σ_2. The corresponding morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is proper if and only if for every $\sigma_2 \in \Sigma_2$ the set*

$$S := \{\sigma_1 \in \Sigma_1 : \psi(\sigma_1) \subseteq \sigma_2\}$$

is finite and

$$\psi^{-1}(\sigma_2) = \bigcup_{\sigma_1 \in S} \sigma_1.$$

A refinement obviously satisfies the second conditions, so we can state:

Lemma 2.1.21. *If in the situation of lemma 2.1.19 the refinement Σ' of Σ is* locally finite*, i.e. every cone of Σ contains only finitely many cones of Σ, the induced morphism $\psi : X_\Sigma \to X_{\Sigma'}$ is proper birational.*

This gives us a rich source of morphisms between toric varieties induced by fans with the same support.
We close our overview of the fundamentals of toric geometry with a look at two geometric invariants of toric varieties. We note that the birational geometry of toric varieties is rather simple: Every toric variety contains by definition a dense torus and is therefore birational to the projectice space of the same dimension. This allows an easy calculation of birational invariants.
Without further definition of the appearing objects we state for later use:

Proposition 2.1.22 ([Ful93, p.75])**.** *Let $X = X_\Sigma$ be a smooth compact toric variety of dimension n. The Euler characteristic $\chi(X, \mathcal{O}_X)$ of the structure sheaf $\mathcal{O}_X$ of X is*

$$\chi(X, \mathcal{O}_X) = 1$$

and its canonical bundle ω_X has Euler characteristic

$$\chi(X, \omega_X) = (-1)^n.$$

We turn to an application of the standard construction of fiber products to the theory of toric varieties.

2.2. Fiber products of toric varieties

We will encounter two different notions of fiber products which differ in general: fiber products in the category of schemes and fiber products in the category of toric varieties, see [Mol16, Section 2.2]. In this section we will introduce the lesser-known latter and explain their relation of these products with each other.
The following definition is the usual notion of fiber products in the category of toric varieties:

Definition 2.2.1. Let T_1, T_2, T_3 be tori with cocharacter lattices $X^*(T_1)$, $X^*(T_2)$, $X^*(T_3)$ and associated fans Σ_1, Σ_2 resp. Σ_3. Assume that there are toric morphisms $F_1 : X_{\Sigma_1} \to X_{\Sigma_3}$ and $f_2 : X_{\Sigma_2} \to X_{\Sigma_3}$. The *toric fiber product* of X_{Σ_1} and X_{Σ_2} over X_{Σ_3} is the toric variety $X_{\Sigma_1} \times_{X_{\Sigma_3}} X_{\Sigma_2}$ together with two toric morphisms $p_1 : X_{\Sigma_1} \times_{X_{\Sigma_3}} X_{\Sigma_2} \to X_{\Sigma_1}$ resp. $p_2 : X_{\Sigma_1} \times_{X_{\Sigma_3}} X_{\Sigma_2} \to X_{\Sigma_2}$ making the diagram

$$\begin{tikzcd}
X_{\Sigma_1} \times_{X_{\Sigma_3}} X_{\Sigma_2} \arrow[r, "p_2"] \arrow[d, "p_1"'] & X_{\Sigma_2} \arrow[d, "f_2"] \\
X_{\Sigma_1} \arrow[r, "f_1"'] & X_{\Sigma_3}
\end{tikzcd}$$

commutative. It is universal in the sense that for any toric variety Q with toric morphisms $q_i : Q \to X_{\Sigma_i}$ $(i = 1, 2)$ there is a unique toric morphism $u : Q \to X_{\Sigma_1} \times_{X_{\Sigma_3}} X_{\Sigma_2}$ with $q_i = p \circ u_i$ for $i = 1, 2$:

$$\begin{tikzcd}
 & Q \arrow[d, dashed, "u"] \arrow[ddl, bend right, "q_1"'] \arrow[ddr, bend left, "q_2"] & \\
 & X_{\Sigma_1} \times_{X_{\Sigma_3}} X_{\Sigma_2} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X_{\Sigma_1} \arrow[dr, "f_1"'] & & X_{\Sigma_2} \arrow[dl, "f_2"] \\
 & X_{\Sigma_3} &
\end{tikzcd}$$

As usual, this fiber product, if it exists, is unique up to unique isomorphism. The existence as well as its construction are given by the following result:

Proposition 2.2.2. *A toric fiber product in the sense of definition 2.2.1 exists. The toric variety $X_{\Sigma_1} \times_{X_{\Sigma_3}} X_{\Sigma_2}$ can be constructed as X_{Σ_0} for Σ_0 a fan in $X^*(T_0)_{\mathbb{R}}$ for a certain torus T_0 with explicit description of all these objects as follows:*

- *The torus T_0 is the fiber product $T_1 \times_{T_3} T_2$ of T_1 and T_2 over T_3 (as abelian groups), so for $T_i \cong (\mathbb{C}^*)^{n_i}$ we have*

$$T_0 \cong (\mathbb{C}^*)^{n_1+n_2-n_3}.$$

- *The group $X^*(T_0)$ and the corresponding real vector space $X^*(T_0)_{\mathbb{R}}$ are isomorphic to $\mathbb{Z}^{n_1+n_2-n_3}$ resp. $\mathbb{R}^{n_1+n_2-n_3}$ (this is also compatible with the aforementioned construction to get tori from lattices and gives the same results).*

- *The fan Σ_0 is*

$$\Sigma_1 \times_{\Sigma_3} \Sigma_2 = \{\sigma_1 \times_{\sigma_3} \sigma_2 \subseteq X^*(T_0) |\, \sigma_i \in \Sigma_i \text{ for } i \in \{1,2,3\}\}$$

with the fiber product $\sigma_1 \times_{\sigma_3} \sigma_2$ of cones σ_1 and σ_2 over σ_3 defined as

$$\sigma_1 \times_{\sigma_3} \sigma_2 = \{(x,y) \in \sigma_1 \times \sigma_2 |\, \phi_{1,\mathbb{R}}(x) = \phi_{2,\mathbb{R}}(y) \in \sigma_3 \subseteq X^*(T_3)_{\mathbb{R}}\} \subseteq X^*(T_0)_{\mathbb{R}}$$

where $\phi_{i,\mathbb{R}}$ denotes the $\mathbb{R}$-linear extension of the compatible map

$$\phi_i : X^*(T_i) \to X^*(T_3)$$

induced by the toric morphism $f_i : X_{\Sigma_i} \to X_{\Sigma_3}$.

Proof. This can be extracted from the construction of toric fiber products given in Definition 2.2.1 of [Mol16]. □

We give one instructive example on how to think of the fiber product of fans:

Example 2.2.3. Let $T_1 = T_2 = (\mathbb{C}^*)^2$ and $T_3 = \mathbb{C}^*$, then $X^*(T_1) = X^*(T_2) \cong \mathbb{Z}^2$ and $X^*(T_3) \cong \mathbb{Z}$. Let Σ_i for $i = 1,2$ be given by the set of cones

$$\{\mathbb{R}_+(m,1) + \mathbb{R}_+((m+1),1) \subseteq X^*(T_i) |\, m \in \mathbb{Z}\}$$

and their faces

$$\{\mathbb{R}_+(m,1) \subseteq X^*(T_i) |\, m \in \mathbb{Z}\}.$$

We'll denote this fan by Σ_{ell}, since it will play a role in the compactification of the universal elliptic curve later on. Let Σ_3 be the trivial fan $\{\{0\}, \mathbb{R}_+\}$ generated by the vector $1 \in \mathbb{Z} \cong X^*(T_3)$. Denote by $f_i : X_{\Sigma_i} \to X_{\Sigma_3} \cong \mathbb{C}$ the toric morphism induced by the compatible map $\phi : \mathbb{Z}^2 \to \mathbb{Z}, (a,b) \mapsto b$. Using the above description of the toric fiber product, we see that $T_0 \cong (\mathbb{C}^*)^3$, $X^*(T_0) \cong \mathbb{Z}^3$ resp. $X^*(T_0)_{\mathbb{R}} \cong \mathbb{R}^3$.

The fan Σ_0 has the following form: Since every non-trivial cone in Σ_i for $i = 1, 2$ has image $\mathbb{R}_+ \in \Sigma_3$, the only possibility for fiber products over $\sigma_3 = \{0\}$ is with

$$\sigma_1 = \sigma_2 = (0, 0) \in \mathbb{R}^2$$

and yields the cone $(0, 0, 0) \in \Sigma_0$. Let $\sigma_i \in \Sigma_i$ for $i \in \{1, 2, 3\}$ be non-trivial, then

$$\begin{aligned} \sigma_1 &= \mathbb{R}_+(m_1, 1) + \mathbb{R}_+(m_1 + 1, 1) \\ \sigma_2 &= \mathbb{R}_+(m_2, 1) + \mathbb{R}_+(m_2 + 1, 1) \end{aligned}$$

for some $m_1, m_2 \in \mathbb{Z}$ and $\sigma_3 = \mathbb{R}_+$, then

$$\begin{aligned} & \sigma_1 \times_{\sigma_3} \sigma_2 \\ \cong & \Big\{[(r_1 m_1 + r_2(m_1 + 1), r_1 + r_2), (s_1 m_2 + s_2(m_2 + 1), s_1 + s_2)] \in \mathbb{R}^2 \times \mathbb{R}^2 : \\ & (r_1, r_2, s_1, s_2) \in (\mathbb{R}_+)^4 \text{ and } \phi_{1,\mathbb{R}}(\ldots) = r_1 + r_2 = s_1 + s_2 = \phi_{2,\mathbb{R}}(\ldots)\Big\} \\ = & \Big\{[(t_1 m_1 + (u - t_1)(m_1 + 1), u), (t_2 m_2 + (u - t_2)(m_2 + 1), u)] \in \mathbb{R}^2 \times \mathbb{R}^2 : \\ & u \in \mathbb{R}_+ \text{ and } (t_1, t_2) \in (0, u)^2\Big\} \\ = & \mathbb{R}_+ \{[(t_1 m_1 + (1 - t_1)(m_1 + 1), 1), (t_2 m_2 + (1 - t_2)(m_2 + 1), 1)] : \\ & (t_1, t_2) \in (0, 1)^2)\Big\} \\ \cong & \mathbb{R}_+ \Big\{(t_1 m_1 + (1 - t_1)(m_1 + 1), t_2 m_2 + (1 - t_2)(m_2 + 1), 1) \in \mathbb{R}^3 : \\ & (t_1, t_2) \in (0, 1)^2\Big\} \\ = & \mathbb{R}_+(m_1, m_2, 1) + \mathbb{R}_+(m_1 + 1, m_2, 1) + \mathbb{R}_+(m_1, m_2 + 1, 1) \\ & +\mathbb{R}_+(m_1 + 1, m_2 + 1, 1). \end{aligned}$$

A similar computation for the cases in which one (or both) of the cones is one-dimensional (a "ray") shows that all the proper faces of the three-dimensional cones $\sigma_1 \times_{\sigma_3} \sigma_2$ from before are contained in Σ_0, so we see that Σ_0 consists exactly of cones of the form

$$\mathbb{R}_+(m_1, m_2, 1) + \mathbb{R}_+(m_1 + 1, m_2, 1) + \mathbb{R}_+(m_1, m_2 + 1, 1) + \mathbb{R}_+(m_1 + 1, m_2 + 1, 1)$$

for $m_1, m_2 \in \mathbb{Z}$ and their faces. We denote this fan by Σ^2_{ell} and call the toric variety corresponding to the above fan the *fiber square* of $X_{\Sigma_{\text{ell}}}$ over $\mathbb{C}$ and denote it by $X^2_{\Sigma_{\text{ell}}}$, so

$$X_{\Sigma^2_{\text{ell}}} = X^2_{\text{ell}} = X_{\Sigma_{\text{ell}}} \times_{\mathbb{C}} X_{\Sigma_{\text{ell}}}.$$

Since obviously $X^2_{\Sigma_{\text{ell}}}$ admits again a toric morphism

$$X^2_{\Sigma_{\text{ell}}} \to \mathbb{C}$$

induced by the compatible mapping $\Sigma_0 \to \{\{0\}, \mathbb{R}_+\}$ one is led to consider higher fiber powers of $X_{\Sigma_{\text{ell}}}$ over $\mathbb{C}$:

Example 2.2.4. Applying the same procedure with the fiber square $X_{\Sigma_{\mathrm{ell}}}^2$ instead of $X_{\Sigma_{\mathrm{ell}}}$ for X_{Σ_1} we can form the fiber cube

$$X_{\Sigma_{\mathrm{ell}}}^3 = X_{\Sigma_{\mathrm{ell}}}^2 \times_{\mathbb{C}} X_{\Sigma_{\mathrm{ell}}} = X_{\Sigma_{\mathrm{ell}}^3}.$$

We can describe the corresponding fan Σ_{ell}^3 quite easily: Note that we can substitute any $m_1 \in \mathbb{Z}$ by $\overrightarrow{m_1} = (m_{1,1}, m_{1,2}) \in \mathbb{Z}^2$ and replace any occurrence of $m_1 + 1$ by the two cases $(m_{1,1}+1, m_{1,2})$ and $(m_{1,1}, m_{1,2}+1)$. Hence the fan Σ_{ell}^3 consists exactly of the cones

$$\sum_{\underline{i} \in \{0,1\}^3} \mathbb{R}_+(\overrightarrow{m} + \underline{i}, 1) \subseteq \mathbb{R}^4$$

for $\overrightarrow{m} \in \mathbb{Z}^3$ and all of its faces.
Of course, the same is true for the higher fiber powers: The fan Σ_{ell}^k consists of the cones

$$\sum_{\underline{i} \in \{0,1\}^k} \mathbb{R}_+(\overrightarrow{m} + \underline{i}, 1) \subseteq \mathbb{R}^{k+1}$$

for $\overrightarrow{m} \in \mathbb{Z}^k$ as well as all of its faces.

Note that these varieties are a priori only fiber products in the category of toric varieties, not necessarily in the category of schemes. In general we can face the following phenomenon:

Example 2.2.5 ([Mol16, Example 2.2.3])**.** Let $T_i = \mathbb{C}^*$ with $\Sigma_i = \{\{0\}, \mathbb{R}_+\}$ for $i = 1, 2, 3$ with the toric morphisms induced by the compatible maps $\phi_1 : \mathbb{Z} \to \mathbb{Z}, a \mapsto 2a$ resp. $\phi_2 : \mathbb{Z} \to \mathbb{Z}, a \mapsto 3a$. The toric fiber product is given simply by $\mathbb{C}$ while the fiber product in the category of schemes is the nodal curve $y^2 = x^3$.

However, there is an explicit condition for the two notions of fiber product to coincide:

Proposition 2.2.6 ([Mol16, Section 2.2])**.** *Given the data for defining the toric fiber product. The toric fiber product $X_{\Sigma_1} \times_{X_{\Sigma_3}} X_{\Sigma_2}$ coincides with the fiber product of schemes if and only if for every cone $\sigma_1 \times_{\sigma_3} \sigma_2 \in \Sigma_1 \times_{\Sigma_3} \Sigma_2$ the map*

$$X^*(T_1)_{\sigma_1}^\vee \oplus_{X^*(T_3)_{\sigma_3}^\vee} X^*(T_2)_{\sigma_2}^\vee \to (\sigma_1 \times_{\sigma_3} \sigma_2)^\vee \cap \Big(X^*(T_1)_{\sigma_1} \times_{X^*(T_3)_{\sigma_3}} X^*(T_2)_{\sigma_2}\Big)^\vee$$

is an isomorphism of semigroups.

To check this for every pair of cones is hard in practice, therefore we will use a sufficient abstract criterion, based on the notion of *weak semistability* (cf. [Mol16, Definition 2.1.2]):

Definition 2.2.7. A toric morphism $\pi : X_{\Sigma_1} \to X_{\Sigma_2}$ is called *weakly semistable* if

- For every cone $\sigma_1 \in \Sigma_1$ there exists $\sigma_2 \in \Sigma_2$ with $\pi(\sigma_1) = \sigma_2$ and
- If $\pi(\sigma_1) = \sigma_2$ for some $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$, then $\pi(\sigma_1 \cap X^*(T_1)) = \sigma_2 \cap X^*(T_2)$.

With this notion one can prove the following result:

Proposition 2.2.8 ([Mol16, Lemma 2.2.6])**.** *Let $\pi_1 : X_{\Sigma_1} \to X_{\Sigma_3}$ be weakly semistable and $\pi_2 : X_{\Sigma_2} \to X_{\Sigma_3}$ be an arbitrary toric morphism. Then*

$$\begin{tikzcd}
X_{\Sigma_1} \times_{X_{\Sigma_3}} X_{\Sigma_2} \arrow[r, "p_2"] \arrow[d, "p_1"'] & X_{\Sigma_2} \arrow[d, "f_2"] \\
X_{\Sigma_1} \arrow[r, "f_1"'] & X_{\Sigma_3}
\end{tikzcd}$$

is a pullback diagram in the category of schemes. In particular $X_{\Sigma_1} \times_{X_{\Sigma_3}} X_{\Sigma_2}$ is the schematic fiber product of X_{Σ_1} and X_{Σ_2} over X_{Σ_3}.

This last statement enables us to prove that in the case of $X^k_{\Sigma_{\mathrm{ell}}}$ the two notions of fiber products agree.

Proposition 2.2.9. *The k-th toric fiber power $X^k_{\Sigma_{ell}}$ of $X^1_{\Sigma_{ell}}$ over $\mathbb{C}$ is also the k-th schematic fiber power of $X^1_{\Sigma_{ell}}$ over $\mathbb{C}$*

Proof. The toric morphism $X_{\mathrm{ell}} \to \mathbb{C}$ induced by the compatible mapping

$$\phi : \mathbb{Z}^2 \to \mathbb{Z}, (a, b) \mapsto b$$

is weakly semistable: This is trivial for the cone $\{0\}$. For non-trivial $\sigma \in \Sigma_{\mathrm{ell}}$ there exists $(x, y) \in \sigma$ with $y > 0$, so $\phi(x, y) = y \in X^*(\mathbb{C}^*)_{\mathbb{R}}$ and hence by homogeneity $\pi(\sigma) = \mathbb{R}_+$ which is a cone in the fan corresponding to $\mathbb{C}$ as a toric variety. A similar reasoning for a preimage of $1 \in \sigma_2 \cap X^*(\mathbb{C}^*)$ shows that $\pi(\sigma_1 \cap X^*(\mathbb{C}^*)) = \mathbb{Z}_{>0} = \sigma_2 \cap X^*(\mathbb{C}^*)$. An application of proposition 2.2.8 shows the case $k = 2$ and repeated use with $X_{\Sigma_2} = X^k_{\mathrm{ell}}$ proves the general case. $\square$

The theory of toric varieties equips us with a functorial way of embedding tori into varieties whose properties are governed by the choice of combinatorial input data. We will use this for the construction of compactifications of locally symmetric spaces in the next chapter.

3. Toroidal compactifications

The theory of toric varieties in the last chapter allows us to present a solution to the problem of compactifying non-compact locally symmetric spaces: the theory of toroidal compactifications, mainly developed by Ash, Mumford, Rapoport and Tai in the excellent book [AMRT10]. A good introduction to the main ideas is [Mum73].
In the first section we will present the general construction of toroidal compactifications with its various intermediate steps and describe the dependence of the resulting space from the input data. The second section deals with the functoriality of this construction, namely with its behavior under closed immersions of the open underlying locally symmetric spaces. The final section will give first examples of this very general theory which will be of particular importance later on.

3.1. General construction

We will follow the treatment in [AMRT10]; other resources used and useful for the exposition are [BZ19] and [Nam09].
Let $\mathcal{D} = G/K$ be a symmetric space of non-compact type and $\Gamma \subseteq G$ an arithmetic subgroup of the real locus of a semisimple algebraic group defined over $\mathbb{Q}$, as in section 1.1. The space $X = \Gamma\backslash\mathcal{D}$ is a non-compact locally symmetric space. Its Baily-Borel compactification $\overline{X}^{\mathrm{BB}}$ is given by the adjunction of Γ-equivalence classes of rational boundary components of $\overline{\mathcal{D}}$ as in theorem 1.1.13.
We recall the realization of $\mathcal{D}$ as a Siegel domain of the third kind: Choose a rational boundary component $\mathcal{F}$ of $\overline{D}$, then

$$\mathcal{D} \cong \{(x, y, z) \in \mathcal{U}(\mathcal{F})_{\mathbb{C}} \times \mathcal{V}(\mathcal{F}) \times \mathcal{F} | \operatorname{Im} x - h_z(y, y) \in \mathcal{C}(\mathcal{F})\}\,.$$

We need to understand and describe the action of Γ in this realization: We define integral versions of the groups appearing in theorem 1.1.16. We set $\mathcal{P}(\mathcal{F})_{\mathbb{Z}} = \Gamma \cap \mathcal{P}(\mathcal{F})$ and similarly $\mathcal{W}(\mathcal{F})_{\mathbb{Z}} = \Gamma \cap \mathcal{W}(\mathcal{F})$, $\mathcal{U}(\mathcal{F})_{\mathbb{Z}} = \Gamma \cap \mathcal{U}(\mathcal{F})$ and $\mathcal{V}(\mathcal{F})_{\mathbb{Z}} = \mathcal{W}(\mathcal{F})_{\mathbb{Z}}/\mathcal{U}(\mathcal{F})_{\mathbb{Z}}$; moreover denote by $\overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$ the image of $\mathcal{P}(\mathcal{F})_{\mathbb{Z}}$ in $G_l(\mathcal{F}) \subseteq \operatorname{Aut}(\mathcal{C}(\mathcal{F}))$, the subgroup of the automorphism group of the cone $\mathcal{C}(\mathcal{F})$ as in proposition 1.1.18.
The quotient

$$\mathcal{D} \to \Gamma\backslash\mathcal{D}$$

can be factored as

$$\mathcal{D} \to \mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D} \to \mathcal{P}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D} \to \Gamma\backslash\mathcal{D}$$

and we can proceed in steps as follows:

1) Compactify $\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D}$ in a way such that the space is modified only locally near $\mathcal{F}$

2) Identify those local compactifications under the action of $\overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$ to get a partial compactification of $\mathcal{P}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D}$

3) Glue all appearing partial compactifications to respect the identification of rational boundary components $\mathcal{F}$ via Γ to cusps of $\overline{X}^{\mathrm{BB}}$

Of course, the details in this are complicated and one has to choose the local compactifications carefully to guarantee an overall compatibility.

Local compactification

The group $\mathcal{U}(\mathcal{F})_{\mathbb{Z}}$ acts via translation on $\mathcal{U}(\mathcal{F})_{\mathbb{C}}$ in the Siegel domain realization

$$\mathcal{D} \cong \{(x,y,z) \in \mathcal{U}(\mathcal{F})_{\mathbb{C}} \times \mathcal{V}(\mathcal{F}) \times \mathcal{F} | \operatorname{Im} x - h_z(y,y) \in \mathcal{C}(\mathcal{F})\},$$

so

$$\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D} \subseteq (\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{U}(\mathcal{F})_{\mathbb{C}}) \times \mathcal{V}(\mathcal{F})_{\mathbb{Z}} \times \mathcal{F}.$$

The crucial point to observe is that

$$\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{U}(\mathcal{F})_{\mathbb{C}} \cong \mathbb{Z}^n\backslash\mathbb{C}^n \cong (\mathbb{C}^*)^n$$

is an algebraic torus via the exponential function, so any toric variety

$$X_\Sigma \supseteq \mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D} \cong (\mathbb{C}^*)^n$$

gives a way of adding points to $\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D}$ by taking its closure inside X_Σ. We denote the interior of the closure

$$\overline{\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D}}^{\circ} \subseteq X_\Sigma$$

of $\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D}$ inside a toric variety X_Σ by

$$(\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D})_\Sigma := \overline{\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D}}^{\circ}.$$

Note that these additional points are near the origin of $(\mathbb{C}^n)^n \subset \mathbb{C}^n$, so their preimage under the exponential map corresponds to points whose imaginary part is at infinity of the cone $\mathcal{C}(\mathcal{F})$. This is the desired locality of step 1).

Partial compactifications

The local compactification process has to be compatible with the reduction step

$$\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D} \to \mathcal{P}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D},$$

via the action of $\overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$; additionally, we want $(\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D})_\Sigma$ to be compact. The corresponding property of the fan Σ defining $(\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D})_\Sigma$ is called $\overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$-*admissibility.* To formulate it, we need the notion of the *rational closure* of $\mathcal{C}(\mathcal{F})$:

Definition 3.1.1. The *rational closure* $\overline{\mathcal{C}(\mathcal{F})}^{\text{rat}}$ is the union of $\mathcal{C}(\mathcal{F})$ with all of its rational boundary components. A rational boundary component of $\mathcal{C}(\mathcal{F})$ is any cone of the form $\mathcal{C}(\mathcal{F}')$ for $\mathcal{F} \subseteq \overline{\mathcal{F}'}$.

We are now able to give the characterization of those fans $\Sigma(\mathcal{F})$ that yield the correct objects $(\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F})}$ for our compactification purpose:

Definition 3.1.2. Let $\Gamma_{\mathcal{F}} \subseteq \mathrm{Aut}(\mathcal{C}(\mathcal{F}))$ be a subgroup. A rational polyhedral partial decomposition $\Sigma(\mathcal{F})$ of $\overline{\mathcal{C}(\mathcal{F})}^{\text{rat}}$ is $\Gamma_{\mathcal{F}}$-*admissible* if

i) $\Sigma(\mathcal{F})$ decomposes $\overline{\mathcal{C}(\mathcal{F})}^{\text{rat}}$: $\bigcup\limits_{\sigma\in\Sigma(\mathcal{F})} \sigma = \overline{\mathcal{C}(\mathcal{F})}^{\text{rat}}$

ii) The decomposition $\Sigma(\mathcal{F})$ is $\Gamma_{\mathcal{F}}$-invariant: For any $\gamma \in \Gamma_{\mathcal{F}}$ and $\sigma \in \Sigma(\mathcal{F})$, the cone $\gamma\sigma$ is also a cone in $\Sigma(\mathcal{F})$.

iii) The decomposition $\Sigma(\mathcal{F})$ is $\Gamma_{\mathcal{F}}$-finite: There are only finitely many classes of cones in $\Sigma(\mathcal{F})$ modulo $\Gamma_{\mathcal{F}}$.

To shorten notation we call such a rational polyhedral partial decomposition simply a $\Gamma_{\mathcal{F}}$-*admissible decomposition of* $\mathcal{C}(\mathcal{F})$. The definition is such that $\Gamma_{\mathcal{F}}$ still acts on $(\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F})}$. The case of interest to us is of course $\Gamma_{\mathcal{F}} = \overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$.
Note that the decomposition and the finiteness conditions are not necessary for the compatibility with $\overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$ but for the overall gluing procedure in the next section. The effect of the condition *i*) will be that the resulting space is indeed compact while *iii*) will result in the toroidal compactification being of finite type.
We formalize the definition of these intermediate objects:

Definition 3.1.3. Given a $\overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$-admissible rational polyhedral cone decomposition $\Sigma(\mathcal{F})$ we call

$$(\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F})} = \overline{\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D}}^{\circ} \subseteq X_{\Sigma(\mathcal{F})}$$

the *partial compactification* at $\mathcal{F}$ with respect to $\Sigma(\mathcal{F})$.

These spaces are of great importance as they yield good local models of the geometry of our toroidal compactification of $\Gamma\backslash\mathcal{D}$.

Toroidal compactification

The preceding section explained how to construct partial compactifications with respect to rational boundary components $\mathcal{F}$ of $\mathcal{D}$. In general, the Γ-equivalence classes of any one of these is not compact and therefore not a compactification of $\Gamma\backslash\mathcal{D}$, so one has to take several (all) of these and glue them appropriately.
Surely one cannot expect an arbitrary collection of partial compactifications to glue together to yield a compactification of the locally symmetric space $\Gamma\backslash\mathcal{D}$. To achieve this, we need the collection of partial compactifications to fulfill some compatibility conditions which we can phrase in the language of the corresponding admissible cone decompositions $\Sigma(\mathcal{F})$:

Definition 3.1.4. A Γ-*admissible family* Σ is a collection of rational polyhedral partial decompositions $\Sigma(\mathcal{F})$ such that for every rational boundary component $\mathcal{F}$ of $\mathcal{D}$ the decomposition $\Sigma(\mathcal{F})$ is $\overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$-admissible and the collection Σ satisfies the compatibility conditions

i) If $\mathcal{F}_2 = \gamma\mathcal{F}_1$ for some $\gamma \in \Gamma$, then $\gamma\Sigma(\mathcal{F}_1) = \Sigma(\mathcal{F}_2)$

ii) If $\mathcal{F}_1 \subseteq \overline{\mathcal{F}_2}$, then $\Sigma(\mathcal{F}_1) = \left\{\sigma \cap \overline{\mathcal{C}(\mathcal{F}_1)}^{\mathrm{rat}} \,|\, \sigma \in \Sigma(\mathcal{F}_2)\right\}$.

Assume now that we have been given a Γ-admissible family. The gluing between the partial compactification is as follows:
If $\mathcal{F}_1, \mathcal{F}_2$ are two rational boundary components of $\mathcal{D}$ with $\mathcal{F}_2 \subseteq \overline{\mathcal{F}_1}$ there is a natural map

$$\pi_{\mathcal{F}_2,\mathcal{F}_1} : (\mathcal{U}(\mathcal{F}_1)_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_1)} \to (\mathcal{U}(\mathcal{F}_2)_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_2)}$$

induced by the inclusion $\mathcal{U}(\mathcal{F}_2) \subseteq \mathcal{U}(\mathcal{F}_1)$ and we can use these maps to define the toroidal compactification $\overline{X}^{\mathrm{tor}}_{\Sigma}$ as the quotient of the disjoint union

$$\bigsqcup_{\substack{\mathcal{F} \text{ rational} \\ \text{boundary component}}} (\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F})}$$

by the equivalence relation $\sim$ given by:

Definition 3.1.5. For $x \in (\mathcal{U}(\mathcal{F}_1)_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_1)}$ and $y \in (\mathcal{U}(\mathcal{F}_2)_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_2)}$ we define $x \sim y$ if and only if there exists a rational boundary component $\mathcal{F}$, $\gamma \in \Gamma$ and $z \in (\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F})}$ with $\mathcal{F}_1, \gamma\mathcal{F}_2 \subseteq \overline{\mathcal{F}}$ and

- z projects to x via the canonical mapping

$$(\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F})} \to (\mathcal{U}(\mathcal{F}_1)_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_1)}$$

- z projects to γy via the canonical mapping

$$(\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F})} \to (\mathcal{U}(\mathcal{F}_2)_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\gamma\mathcal{F}_2)}\,.$$

It is not immediately clear that this relation is indeed transitive: A proof can be found in [AMRT10, III.5, Lemma 5.5].
The following theorem summarizes the most important features of the resulting space:

Theorem 3.1.6 ([AMRT10, Theorem 5.2])**.** *Let Σ be a Γ-admissible family. The toroidal compactification $\overline{X}^{tor}_{\Sigma}$ is the unique Hausdorff analytic variety containing $X = \Gamma\backslash\mathcal{D}$ as an open dense subset such that, for every boundary component $\mathcal{F}$ of $\mathcal{D}$, there is an analytic open morphism $\pi_{\mathcal{F}}$ making the following diagram commutative:*

$$\begin{tikzcd}
\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D} \arrow[r, hook] \arrow[d] & (\mathcal{U}(\mathcal{F})_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F})} \arrow[d, dashed, "\pi_F"] \\
\Gamma\backslash\mathcal{D} \arrow[r, hook] & \overline{\Gamma\backslash\mathcal{D}}^{tor}_{\Sigma}
\end{tikzcd}$$

Moreover, the space $\overline{X}^{tor}_{\Sigma}$ is compact and algebraic; it is the normalization of a blow-up of $\overline{X}^{BB}$ in a certain ideal sheaf $\mathcal{I}_m$.

The name *toroidal compactification* is obviously due to the fact that the resulting space is locally *torus-like* or *toroidal*.
The importance of toroidal compactifications comes from several facts: Firstly, the useful features of toric geometry, i.e. the characterization of geometric properties by combinatorial properties, have analogues for toroidal compactifications. Secondly, as we will see later on in chapter 12, toroidal compactifications behave very well with respect to the extension of vector bundles.
We list some of the phenomena that toroidal compactifications inherit from toric geometry: Since toroidal compactifications look locally like toric varieties, the local properties can be checked on the admissible family.

Lemma 3.1.7. *For neat Γ the toroidal compactification $\overline{X}^{tor}_{\Sigma}$ is smooth if, for every rational boundary component $\mathcal{F}$, every cone in $\Sigma(\mathcal{F})$ is generated by a part of a basis of the lattice $\mathcal{U}(\mathcal{F})_{\mathbb{Z}}$.*

Somewhat surprisingly, projectivity can also be checked in terms of the combinatorial data Σ:

Lemma 3.1.8. *Let Γ be a neat subgroup. The toroidal compactification $\overline{X}^{tor}_{\Sigma}$ is projective if the Γ-admissible family Σ is projective, i.e. for every rational boundary component $\mathcal{F}$ there exists a continuous convex $\overline{\mathcal{P}(\mathcal{F})_{\mathbb{Z}}}$-invariant piecewise linear function $\phi_{\mathcal{F}} : \overline{\mathcal{C}(\mathcal{F})}^{rat} \to \mathbb{R}$ satisfying*

i) $\phi_{\mathcal{F}}$ *is integral on* $\overline{\mathcal{C}(\mathcal{F})}^{rat} \cap \mathcal{U}(\mathcal{F})_{\mathbb{Z}}$

ii) $\phi_{\mathcal{F}}(x) > 0$ *for* $x \neq 0$

iii) $\phi_{\mathcal{F}}$ *is linear on a closed polyhedral cone* $\sigma \subseteq \overline{\mathcal{C}(\mathcal{F})}^{rat}$ *if and only if* $\sigma \subseteq \sigma'$ *for some* $\sigma' \in \Sigma(\mathcal{F})$.

Moreover, the collection $\{\phi_{\mathcal{F}} | \mathcal{F}$ rational boundary component$\}$ has to be compatible in the sense that

i) If $\mathcal{F}_2 = \gamma\mathcal{F}_1$ for some $\gamma \in \Gamma$, we have $\phi_{\mathcal{F}_1} = \phi_{\mathcal{F}_2} \circ \gamma$

ii) If $\mathcal{F}_2 \subseteq \overline{\mathcal{F}_1}$, then $\phi_{\mathcal{F}_2}|_{\overline{\mathcal{C}(\mathcal{F}_1)}^{rat}} = \phi_{\mathcal{F}_1}$.

Toroidal compactifications have a very well-behaved boundary: For certain choices of Σ it consists of simple normal crossing divisors whose definition we give for reference.

Definition 3.1.9. A Weil divisor $D = \sum_i D_i \subseteq X$ on a smooth variety X of dimension n is *normal crossing* if any component D_i is smooth and for every point there is a local equation of D of the form

$$x_1 \cdot \ldots \cdot x_k = 0$$

with local analytic coordinates $x_1, \ldots, x_k$. The divisor is called *simple normal crossing* if we have $k \leq n$ at all points.

There are the additional related resp. synonymous notions of *strict normal crossing divisors* and *smooth normal crossing divisors* which sometimes share the unfortunate abbreviation *SNC divisor*.
If Σ is smooth, the boundary divisor of a toroidal compactification is a normal crossing divisor. To get a simple normal crossing divisor, one needs Σ to satisfy an additional condition:

Lemma 3.1.10. *Suppose that Γ is neat and the Γ-admissible family Σ of cone decompositions is smooth and satisfies the following condition:*
If $\gamma \in \overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$ satisfies $\gamma(\sigma) \cap \sigma \neq \{0\}$ for a cone $\sigma \in \Sigma(\mathcal{F})$ then a power of γ acts as the identity on the smallest boundary component $\mathcal{C}(\mathcal{F}') \subseteq \overline{\mathcal{C}(\mathcal{F})}^{rat}$ containing σ.
Then: The boundary $\overline{X}_{\Sigma}^{tor} \setminus X$ is a simple normal crossing divisor.

Proof. This is proved as in [YZ14]. Note, however, that the reasoning in [YZ14] has a subtle error: The correct definition of Γ-separability should be (in the local notation)

> $(\ldots)$ if a $\gamma \in \overline{\Gamma_{\mathfrak{F}}}$ satisfies $\gamma(\sigma) \cap \sigma \neq \{0\}$ for a cone $\sigma \in \Sigma_{\mathfrak{F}}$ then a power of γ acts as the identity on the *minimal admissible boundary component* $\mathfrak{F}'$ containing this intersection.

Here an *admissible boundary component* is the image of an embedding $\mathcal{C}_{\mathfrak{F}'} \hookrightarrow \overline{\mathcal{C}_{\mathfrak{F}}}$ for some adjacent boundary component $\mathfrak{F}'$ of $\mathfrak{F}$. □

With this corrected definition, the result in [YZ14, Theorem 2.22] is in accordance with the analogous statements [Pin89, MP11]. There seemed to be a thorough confusion about this point in the literature since similar mistakes about this property appear in [FC90, Chapter IV, Remark 5.8] and older versions of [Lan08, Condition 6.2.5.25].

Remark 3.1.11. As a toroidal compactification $\overline{X}_{\Sigma}^{\mathrm{tor}}$ of X contains X as an open dense subset, any two of these compactifications are birational. If for a toroidal compactification $\overline{X}_{\Sigma'}^{\mathrm{tor}}$ the defining admissible family Σ' is a sufficiently fine (smooth) *refinement* of the defining admissible family Σ of $\overline{X}_{\Sigma}^{\mathrm{tor}}$, the resulting morphism $\overline{X}_{\Sigma'}^{\mathrm{tor}} \to \overline{X}_{\Sigma}^{\mathrm{tor}}$ is projective (see [AMRT10, Chapter III, Corollary 7.6]), thus proper.

As a concluding remark for the general construction we claim that the last two condition are not too strict: Every Γ-admissible family can be refined as to be smooth and projective. The existence of Γ-admissible families is secured by the abstract theory of *cores* and *co-cores* as in [AMRT10, Chapter 2]. Note that this construction is rather indirect and only mildly constructive.

3.2. Functoriality

Having described the properties of a single toroidal compactification, we want to describe the relation of toroidal compactifications for symmetric spaces $\mathcal{D}_1 \hookrightarrow \mathcal{D}_2$. For general unrelated $\Gamma_i \subset \mathrm{Aut}(\mathcal{D}_\mathrm{i})$ one cannot expect any meaningful relation between $\Gamma_1 \backslash \mathcal{D}_1$ and $\Gamma_2 \backslash \mathcal{D}_2$.
This is different if $\Gamma_1 = \Gamma_2 \cap \mathrm{Aut}(\mathcal{D}_1)$:

Lemma 3.2.1. *Let $\mathcal{D}_1 \hookrightarrow \mathcal{D}_2$ be an inclusion of symmetric spaces and Γ_2 be an arithmetic subgroup of* $\mathrm{Aut}(\mathcal{D}_2)$. *Let $\Gamma_1 = \mathrm{Aut}(\mathcal{D}_1) \cap \Gamma_2$. There is a canonical morphism*

$$\psi_{BB} : \overline{\Gamma_1 \backslash \mathcal{D}_1}^{BB} \to \overline{\Gamma_2 \backslash \mathcal{D}_2}^{BB}$$

making the diagram

$$\begin{tikzcd} \Gamma_1 \backslash \mathcal{D}_1 \arrow[r, "\psi"] \arrow[d, hook] & \Gamma_2 \backslash \mathcal{D}_2 \arrow[d, hook] \\ \overline{\Gamma_1 \backslash \mathcal{D}_1}^{BB} \arrow[r, "\psi_{BB}"] & \overline{\Gamma_2 \backslash \mathcal{D}_2}^{BB} \end{tikzcd}$$

commutative. Moreover, for any cusp F_1 of $\overline{\Gamma_1 \backslash \mathcal{D}_1}^{BB}$, there exists a unique cusp F_2 with $\psi_{BB}(F_1) \subseteq F_2$. On the level of symmetric spaces we have the following: Let $\mathcal{F}_2$ be any rational boundary component of $\mathcal{D}_2$ and let $\mathcal{F}_1$ a rational boundary component of $\mathcal{D}_1$ such that the corresponding cusps F_1, F_2 obey $\psi_{BB}(F_1) \subseteq F_2$: There are canonical embeddings

- *$\mathcal{P}(\mathcal{F}_1) \hookrightarrow \mathcal{P}(\mathcal{F}_2)$ of the corresponding maximal rational parabolics,*
- *$\mathcal{U}(\mathcal{F}_2) \hookrightarrow \mathcal{U}(\mathcal{F}_1)$ of the centers of their unipotent radicals as well as*
- *$\iota : \mathcal{C}(\mathcal{F}_1) \hookrightarrow \mathcal{C}(\mathcal{F}_2)$ of the self-adjoint cones $\mathcal{C}(\mathcal{F}_i)$ contained in $\mathcal{U}(\mathcal{F}_i)$ which appear in the respective realization of $\mathcal{D}_i$ as a Siegel domain of the third type.*

Proof. This can be found in this formulation in [Lan19, Proposition 3.4] or originally in [Har89, Section 3]. □

Since we are now working mainly on the locally symmetric spaces, we will adopt the following notation: Let $\mathcal{F}$ be a rational boundary component of $\mathcal{D}$ and $F = \Gamma\mathcal{F}$ the corresponding cusp of $\Gamma\backslash\mathcal{D}$: For any object $\mathcal{O}(\mathcal{F})$ related to the maximal parabolic $\mathcal{P}(\mathcal{F})$ we will write $\mathcal{O}(F)$.
Using this framework, we can generalize this situation to toroidal compactifications. For a given Γ_2-admissible family we can find a Γ_1-admissible family with the following property:

Proposition 3.2.2. *Let $\mathcal{D}_1 \hookrightarrow \mathcal{D}_2$ be an inclusion of symmetric spaces and Γ_2 be an arithmetic subgroup of* $\mathrm{Aut}(\mathcal{D}_2)$. *Let $\Gamma_1 = \mathrm{Aut}(\mathcal{D}_1) \cap \Gamma_2$ and*

$$\psi : \Gamma_1 \backslash \mathcal{D}_1 \to \Gamma_2 \backslash \mathcal{D}_2$$

be the natural morphism. Let Σ_2 be a Γ_2-admissible family of cone decompositions corresponding to a given toroidal compactification $\overline{\Gamma_2 \backslash \mathcal{D}_2}^{tor}_{\Sigma_2}$.
Then there is an Γ_1-admissible family Σ_1 of cone decompositions yielding a toroidal compactification $\overline{\Gamma_1 \backslash \mathcal{D}_1}^{tor}_{\Sigma_1}$ and a natural morphism

$$\psi_{tor} : \overline{\Gamma_1 \backslash \mathcal{D}_1}^{tor}_{\Sigma_1} \to \overline{\Gamma_2 \backslash \mathcal{D}_2}^{tor}_{\Sigma_2},$$

depending on Σ_1 and Σ_2, making the diagram

$$\begin{tikzcd} \Gamma_1\backslash\mathcal{D}_1 \arrow[r, "\psi"] \arrow[d, hook] & \Gamma_2\backslash\mathcal{D}_2 \arrow[d, hook] \\ \overline{\Gamma_1\backslash\mathcal{D}_1}^{tor}_{\Sigma_1} \arrow[r, "\psi_{tor}"] & \overline{\Gamma_2\backslash\mathcal{D}_2}^{tor}_{\Sigma_2} \end{tikzcd}$$

commutative.

Proof. This is again in [Lan19, Proposition 3.4] and [Har89, Section 3]: The main idea is to construct an admissible family for $\Gamma_1\backslash\mathcal{D}_1$ by, locally for each cusp F_1 of $\Gamma_1\backslash\mathcal{D}_1$, simply intersecting any cone of $\Sigma(F_2)$ with the cone $\mathcal{C}(F_1)$ to get a cone decomposition $\Sigma(F_1)$ of $\mathcal{C}(F_1)$. □

Note that this does not give any properties of the morphism ψ_{tor} beyond its mere existence. It is to be expected that its properties depend on the relation of the admissible families as these determine the toroidal compactification. The following three notions of compatibility of admissible families characterize certain properties of the morphism ψ_{tor} between the toroidal compactifications:

Definition 3.2.3 ([Lan19, Proposition 3.4 (4) and Definition 4.5])**.** Two Γ_1- resp. Γ_2-admissible families $\Sigma_1 = \{\Sigma(F_1)\}_{F_1}$ resp. $\Sigma_2 = \{\Sigma(F_2)\}_{F_2}$ of cone decompositions are *compatible with each other* or, shorter, *compatible* if for any embedding $\mathcal{P}(F_1) \hookrightarrow \mathcal{P}(F_2)$ as in lemma 3.2.1 and any cone $\sigma \in \Sigma(F_1)$ (so $\sigma \subseteq \mathcal{C}(F_1)$) there exists a $\tau \in \Sigma(F_2)$ with $\sigma \hookrightarrow \tau$ under the embedding $\mathcal{C}(F_1) \hookrightarrow \mathcal{C}(F_2)$.
The admissible family Σ_1 is *induced by* Σ_2 if every $\sigma \in \Sigma(F_1)$ is of the form $\iota^{-1}(\tau)$ for some $\tau \in \Sigma(F_2)$. Lastly, Σ_1 and Σ_2 are said to be *strictly compatible with each other* (or *strictly compatible*) if $\iota(\sigma) \in \Sigma(F_2)$ for any $\sigma \in \Sigma(F_1)$.

The implications between these notions are

$$\text{strictly compatible} \Rightarrow \text{induced} \Rightarrow \text{compatible.}$$

We are now able to state the following result due to Lan which completely characterizes the properties of the morphism between the toroidal compactifications in terms of the admissible families:

Theorem 3.2.4 ([Lan19, Proposition 3.4 (6) and Proposition 4.8])**.** *Let the natural morphism*

$$\psi : \Gamma_1\backslash\mathcal{D}_1 \to \Gamma_2\backslash\mathcal{D}_2$$

be finite. Then:

- *The admissible collections Σ_1 and Σ_2 are compatible if and only if ψ_{tor} is a proper morphism.*
- *Σ_1 is induced by Σ_2 if and only if ψ_{tor} is a finite morphism.*

If $\psi : \Gamma_1\backslash\mathcal{D}_1 \to \Gamma_2\backslash\mathcal{D}_2$ *is a closed immersion and* Σ_1, Σ_2 *are strictly compatible with each other, the morphism* $\psi_{tor} : \overline{\Gamma_1\backslash\mathcal{D}_1}^{tor}_{\Sigma_1} \to \overline{\Gamma_2\backslash\mathcal{D}_2}^{tor}_{\Sigma_2}$ *is a closed immersion, extending* ψ*.*

The result of Harris in proposition 3.2.2 is just the case of Σ_1 being induced by Σ_2 and shows how to construct Σ_1 from a given Σ_2.
Many of the properties of Σ_2 are inherited by a strictly compatible Σ_1:

Proposition 3.2.5. *Let* $\psi : \Gamma_1\backslash\mathcal{D}_1 \to \Gamma_2\backslash\mathcal{D}_2$ *be a closed immersion and* Σ_1, Σ_2 *strictly compatible with each other. If* Σ_2 *is projective and smooth, so is* Σ_1*.*

Smoothness can be shown by a simple local calculation. The projectivity follows by the restriction of the corresponding projective functions as in lemma 3.1.8.
It is not obvious that proposition 3.2.5 is not an empty statement, since the existence of $\Gamma_i, \mathcal{D}_i$ with $\psi : \Gamma_1\backslash\mathcal{D}_1 \to \Gamma_2\backslash\mathcal{D}_2$ a closed immersion, or of strictly compatible Σ_i is not at all obvious. The next results by Deligne and Hörmann show that there are plenty of examples of closed immersions between locally symmetric spaces as in theorem 3.2.4.

Proposition 3.2.6. *Let* $\mathcal{D}_1 \subseteq \mathcal{D}_2$ *an inclusion of symmetric spaces and* $\Gamma_1 \subseteq \mathrm{Aut}(\mathcal{D}_1)$ *be an arithmetic subgroup.*

(i) There is an arithmetic subgroup $\Gamma_2 \subseteq \mathrm{Aut}(\mathcal{D}_2)$ *with* $\Gamma_1 \subseteq \Gamma_2$ *such that*

$$\Gamma_1\backslash\mathcal{D}_1 \to \Gamma_2\backslash\mathcal{D}_2$$

is a closed immersion.

(ii) Given a closed embedding $\Gamma_1\backslash\mathcal{D}_1 \to \Gamma_2\backslash\mathcal{D}_2$ *for a neat* Γ_2*, for any* $\Gamma_2' \subseteq \Gamma_2$ *the morphism*

$$\Gamma_1'\backslash\mathcal{D}_1 \to \Gamma_2'\backslash\mathcal{D}_2$$

for $\Gamma' = \Gamma_2' \cap \mathrm{Aut}(\mathcal{D}_1)$ *is a closed embedding as well.*

(iii) For an orthogonal symmetric space coming from a lattice as in our case (cf. section 6.2) there is an integer N *such that the congruence subgroup* $\Gamma(N) \subseteq \mathrm{Aut}(\mathcal{D}_2)$ *can be chosen in case (ii).*

Proof. Parts (i) and (ii) can be found in [Del71b, Proposition 1.15] or in the thesis [Pin89, Chapter 9]. Part (iii) is [Hör10, Lemma 2.2.8] □

The existence of sufficiently many strictly compatible families with special properties is guaranteed by the result of Lan:

Proposition 3.2.7. *Given a finite number of admissible families for* $\Gamma_1\backslash\mathcal{D}_1$ *and* $\Gamma_2\backslash\mathcal{D}_2$*, there exist strictly compatible, smooth and projective admissible families* Σ_1 *and* Σ_2 *refining all of the given admissible families simultaneously.*

Proof. This is [Lan19, Proposition 4.9] and is obtained by inductively refining all of the cones of the given admissible family Σ_2 and the induced admissible family Σ_1. □

Summarizing, we see that for a suitably choice of an Γ_2-admissible family Σ_2, the closed embedding

$$\Gamma_1\backslash\mathcal{D}_1 \hookrightarrow \Gamma_2\backslash\mathcal{D}_2$$

can be naturally extended to a closed immersion

$$\overline{\Gamma_1\backslash\mathcal{D}_1}^{\text{tor}}_{\Sigma_1} \hookrightarrow \overline{\Gamma_2\backslash\mathcal{D}_2}^{\text{tor}}_{\Sigma_2}$$

with Σ_1 determined by Σ_2.
Note that, even though we presented the theory of toroidal compactifications solely as a feature of locally symmetric spaces, this also works on the level of Shimura varieties and has been turned arithmetic in some cases, see the work of Lan [Lan08], Hörmann [Hör10] and Madapusi-Pera [MP11]. This fact will be of no further importance for us.

3.3. Examples

It is useful to see a few examples of this rather general theory.
The simplest example is the case of the modular curve $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} = \Gamma\backslash\mathcal{D}$ of dimension 1 which we considered as a locally symmetric space in example 1.1.3.
There is a unique zero-dimensional cusp $F = i\infty$ and the Siegel domain realization turns out to be trivial since the unipotent radical with respect to this cusps is $\mathcal{U}(F) \cong \mathbb{R}$ and $\mathcal{V}(F) = \{0\}$, so $\mathcal{C}(F) = \mathbb{R}_{>0}$ and hence:

$$\begin{aligned} \mathcal{D} &\cong \{(u,\tau,z) \in \mathcal{U}(F)\otimes\mathbb{C} \times F \times \mathcal{V}(F)) : \Im(u) \in \mathcal{C}(F) + h(\tau,z)\} \\ &\cong \{u \in \mathbb{C} : \mathrm{Im}(u) > 0\} \end{aligned}$$

In particular, the torus $\mathcal{T}(F)$ is

$$\mathcal{T}(F) \cong \mathcal{U}(F)\otimes\mathbb{C}/\mathcal{U}(F)_{\mathbb{Z}} \cong \mathbb{C}^*$$

and the cone $\mathcal{C}(F) \cong \mathbb{R}_{>0}$ has the unique trivial decomposition $\Sigma(F) = \{\mathbb{R}_{>0}\}$; hence the toroidal compactification of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ is obtained by gluing it with

$$(\mathcal{U}(F)_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(F)} \cong \overline{\Delta^\circ} \subseteq (\mathbb{C}^*)_{\mathbb{R}_{>0}} = \mathbb{C}.$$

This just fills in the missing zero of the open punctured disc Δ° and yields the usual (Baily-Borel) compactification of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.
In general, toroidal compactifications are not canonical, as we will see in the next example.

Compactifications of Kuga-Sato varieties: classical and toroidal

We give a short survey of another one of the simplest (albeit, for our purposes, most important) examples of toroidal compactifications; this case shows most of the essential features and objects appearing in the general theory.

We will first present it in the classic and constructive 'scissor-and-glue'-language of [AMRT10, Chapter 1, section 4] which we follow closely but generalize their construction to higher rank lattices. Afterwards we will recast in the language of toroidal compactifications we developed in the preceding section. These two differing points of view will prove useful in later chapters.
Let L be any rank n lattice (read here as: free $\mathbb{Z}$-module of rank n, so no quadratic form), $N > 3$ an integer and let $\Gamma(N)$ be the N-th principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, that is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | a \equiv d \equiv 1 \bmod N, b \equiv c \equiv 0 \bmod N \right\}.$$

Its action (and more generally, of $\mathrm{SL}_2(\mathbb{Z})$) on the upper half-plane $\mathbb{H}$ by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).\tau = \frac{a\tau + b}{c\tau + d}$$

is well known and (for any subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$) can be generalized to an action of the affine group

$$\Gamma^A = \Gamma \ltimes L^2$$

with the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.(m, n) = (m, n) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (am + cn, bm + dn)$$

on

$$\mathbb{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C}) = \mathbb{H} \times \mathbb{C}^n$$

by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (m, n) \right).(\tau, \overrightarrow{z}) = \left(\frac{a\tau + b}{c\tau + d}, \frac{\overrightarrow{z} + \tau m + n}{c\tau + d} \right).$$

The following is well-known:

Lemma 3.3.1. *The quotient $Y(N) := \Gamma(N)\backslash\mathbb{H}$ is the moduli space of level-N elliptic curves and*

$$\mathcal{E}^{(n)} = \Gamma(N)^A \backslash \left(\mathbb{H} \times \mathbb{C}^n\right)$$

is the n-fold fiber product of the universal level-N elliptic surface with itself over the modular curve $Y(N)$.

This is often called the *Kuga-Sato variety* of rank n over $Y(N)$. Note that

$$\Gamma(N)^A \subseteq \mathrm{Aut}(\mathbb{H} \times \mathbb{C}^n)$$

is an arithmetic subgroup and

$$\mathcal{E}^{(n)} = \Gamma(N)^A \backslash (\mathbb{H} \times \mathbb{C}^n)$$

is a locally symmetric space. It comes with a natural map $\mathcal{E}^{(n)} \twoheadrightarrow Y(N)$. Due to $N > 3$, the group $\Gamma(N)$ is neat and hence $\mathcal{E}^{(n)}$ and $Y(N)$ are smooth.
The Baily-Borel compactification of $Y(N)$ is obtained by adjoining $\mathbb{P}^1(\mathbb{Q})$ to $\mathbb{H}$ before quotienting; this gives a smooth compactification $\overline{Y(N)}$ of $Y(N)$ with finitely many cusps added and the problem is to find $\overline{\mathcal{E}^{(n)}}$ such that the diagram

$$\begin{tikzcd} \mathcal{E}^{(n)} \arrow[r, hook] \arrow[d] & \overline{\mathcal{E}^{(n)}} \arrow[d] \\ Y(N) \arrow[r, hook] & \overline{Y(N)} \end{tikzcd}$$

commutes. By the transitivity of $\mathrm{SL}_2(\mathbb{Z})$ on the cusps and their discreteness it suffices to construct the part of the compactification $\overline{\mathcal{E}^{(n)}}$ lying over one cusp and translate it to the others, so we can concentrate on the cusp $i\infty = [1:0] \in \overline{Y(\Gamma(N))}$:
We note that for sufficiently large imaginary part the action of $\Gamma(N)$ on $\mathbb{H}$ reduces to the action of the group generated by $T^N = \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$. This gets inherited to the action of $\Gamma(N)^A$ on $\mathbb{H} \times \mathbb{C}^n$ as follows:

Corollary 3.3.2 ([AMRT10], Chapter 1.4, Corollary 4.2)**.** *For $d_0 \gg 0$ let $\mathbb{H}_{d_0}$ be the subset of $\mathbb{H}$ with imaginary part greater than d_0. For $(\tau, \overrightarrow{z}) \in \mathbb{H} \times \mathbb{C}^n$ and $\gamma \in \Gamma(N)^A$ one finds:*

$$(\tau, \overrightarrow{z}) \text{ and } \gamma(\tau, \overrightarrow{z}) \in \mathbb{H}_{d_0} \times \mathbb{C}^n$$

implies that $\gamma \in \Gamma(N)^A_2$, the subgroup of $\Gamma(N)^A$ generated by

$$\{(T^N, (0,0)), (\mathrm{id}, (e_1, 0)), \ldots, (\mathrm{id}, (e_n, 0)), (\mathrm{id}, (0, e_1)), \ldots (\mathrm{id}, (0, e_n))\}$$

with $e_1, \ldots, e_n$ a basis of L.

Let $e_1, \ldots, e_n$ be a fixed basis of L. We factor the projection map

$$p : \mathbb{H} \times \mathbb{C}^n \to \Gamma(N)^A \backslash (\mathbb{H} \times \mathbb{C}^n)$$

using the exponential map $\overrightarrow{\exp}$ given by $\tau \mapsto e^{\frac{2\pi i \tau}{N}}$ and

$$\overrightarrow{z} = \sum_i \lambda_i e_i \mapsto \left(e^{2\pi i \lambda_i}\right)_{1,\ldots,n}.$$

This gives

$$\overrightarrow{\exp}(\mathbb{H} \times \mathbb{C}^n) = \Delta^* \times (\mathbb{C}^*)^n \cong (\mathbb{H} \times \mathbb{C}^n) / \ker(\overrightarrow{\exp}) = (\mathbb{H} \times \mathbb{C}^n) / \Gamma(N)^A_1$$

where $\Gamma(N)^A_1$ is the subgroup of $\Gamma(N)^A$ generated by

$$\{(T^N, (0,0)), (\mathrm{id}, (0, e_1)), \ldots (\mathrm{id}, (0, e_n)).$$

Obviously, corollary 3.3.2 implies now that we get an injection/inclusion

$$\begin{aligned}
\left(\Delta_{d_0}^* \times (\mathbb{C}^*)^n\right) / \left(\Gamma(N)_2^A / \Gamma(N)_1^A\right) &\cong \left(\left(\mathbb{H}_{d_0} \times \mathbb{C}^n\right) / \Gamma(N)_1^A\right) / \left(\Gamma(N)_2^A / \Gamma(N)_1^A\right) \\
&\cong \left(\mathbb{H}_{d_0} \times \mathbb{C}^n\right) / \Gamma(N)_2^A \\
&= \left(\mathbb{H}_{d_0} \times \mathbb{C}^n\right) / \Gamma(N)^A \\
&\subseteq \left(\mathbb{H} \times \mathbb{C}^n\right) / \Gamma(N)^A \\
&= \mathcal{E}^{(n)}
\end{aligned}$$

We give a better description of the group $\Gamma(N)_0^A := \Gamma(N)_2^A / \Gamma(N)_1^A$ acting on $\Delta_{d_0}^* \times (\mathbb{C}^*)^n$: It is isomorphic to the (free) group generated by

$$(\mathrm{id}, (e_1, 0)), \ldots, (\mathrm{id}, (e_n, 0)),$$

hence isomorphic to $\mathbb{Z}^n$.
With the general construction of toroidal compactifications in mind, it is obvious how to proceed from here:
Choose a fan Σ such that X_Σ is a toric variety

$$X_\Sigma \supseteq \mathbb{C}^* \times (\mathbb{C}^*)^n \supseteq \Delta_{d_0}^* \times (\mathbb{C}^*)^n$$

respecting the action of $\Gamma(N)_0^A$ and denote by P the interior of the closure of $\Delta_{d_0}^* \times (\mathbb{C}^*)^n$ in X_Σ. For suitable choices of X_Σ the quotient of P by $\Gamma(N)_0^A$ is a smooth manifold (since the group $\left(\Gamma(N)_2^A / \Gamma(N)_1^A\right)$ acts discontinuously on P) can be glued with $\mathcal{E}^{(n)}$ on the common open subset $\left(\Delta_{d_0} \times (\mathbb{C}^*)^n\right) / \Gamma(N)_0^A$ to get a smooth manifold resolving the cusp $i\infty$ (i.e. proper over Δ_{d_0}).
The transitivity of $\mathrm{SL}_2(\mathbb{Z})$ allows us to move this construction to all cusps and hence get a smooth compactification $\overline{X}$ of $\mathcal{E}^{(n)}$.
As usual we can state conditions sufficient for our construction to work in the language of cones and cone decompositions:

Proposition 3.3.3. *Denote the interior of the closure of $\Delta_{d_0}^* \times (\mathbb{C}^*)^n$ in X_Σ by P. The quotient $P/\Gamma(N)_0^A$ is a smooth manifold and proper over Δ_{d_0} if*

i) the fan Σ is smooth,

ii) the support $|\Sigma|$ of Σ is the set $\mathbb{R}_+ \times \mathbb{R}^n$, and

iii) the fan Σ is invariant under the action of $\Gamma(N)_0^A$ and has only finitely many equivalence classes.

Proof. The smoothness of Σ implies the smoothness of X_Σ resp P by lemma 2.1.9. The group $\Gamma(N)_0^A$ acts properly discontinuously on P by [AMRT10, Chapter III, Proposition 6.10], hence $P/\Gamma(N)_0^A$. The proof there depends on the proof of [AMRT10, Chapter III, Theorem 1.4] which makes heavy use of properties ii) and iii); the same proof also shows that $P/\Gamma(N)_0^A$ is proper over Δ_{d_0} by sequential compactness of preimages of compact sets in Δ_{d_0}. $\square$

We recast this in the language of toroidal compactifications:
The Baily-Borel compactification $\overline{Y}^{\mathrm{BB}}$ of $Y = Y(N)$ is obtained by adjoining the rational points $\mathbb{P}^1(\mathbb{Q}) \cup \{i\infty\}$ to $\mathbb{H}$, so the boundary of Y consists of finitely many zero-dimensional cusps. Let F be a cusp of Y. Up to conjugation by $\mathrm{SL}_2(\mathbb{Z})$ we can assume it arises from $\mathcal{F} = i\infty$. The Siegel domain realization of Y with respect to $i\infty$ is

$$Y \cong \Big\{(x, y, z) \in \mathbb{C}^{n+1} \times \{0\} \times \{i\infty\} \big| \operatorname{Im} x_1 \in \mathbb{R}_{>0}\Big\}$$

with $\mathcal{U}(i\infty)_{\mathbb{C}} \cong \mathbb{C}^{n+1}$, $\mathcal{V}(i\infty) \cong \{0\}$ and $\mathcal{C}(i\infty) \cong \mathbb{R}_{>0} \times \mathbb{R}^n$. The parabolic $\mathcal{P}(i\infty)$ is given by $\Gamma(N)_2^A$ while $\Gamma(N)_0^A$ corresponds to $\overline{\mathcal{P}(i\infty)}$; the torus $\mathcal{T}(i\infty)$ is $(\mathbb{C}^*)^{n+1}$. The interior P of the closure of $\mathcal{U}(i\infty)\backslash\mathcal{D}$ in X_Σ is simply the partial compactification in the sense of definition 3.1.3; lastly, the group $\Gamma(N)_1^A$ is just $\mathcal{U}(i\infty)_{\mathbb{Z}}$. The conditions on the fan in proposition 3.3.3 are exactly those of definition 3.1.2 for a smooth fan.
Note that the compatibility conditions in definition 3.1.4 are trivially satisfied: By construction (i.e. the translation by $\mathrm{SL}_2(\mathbb{Z})$) property i) holds; property ii) does not apply as all of the cusps are zero-dimensional and hence isolated.
We formulate this as a result. Note that L acts on $\mathbb{R}_{>0} \times \mathbb{R}^n$ via

$$\overline{\mathcal{P}(i\infty)} \cong L.$$

Lemma 3.3.4. *Let L be a lattice of rank n. Let $\Sigma(i\infty)$ be a smooth L-admissible decomposition of*

$$\mathcal{C}(i\infty) \cong \mathbb{R}_{>0} \times \mathbb{R}^n$$

and define a $\Gamma^A = \mathrm{SL}_2(\mathbb{Z}) \ltimes L^2$-admissible family $\Sigma = \{\Sigma(\mathcal{F}) | \mathcal{F}$ cusp$\}$ by

$$\Sigma(\mathcal{F}) := \gamma\Sigma(i\infty)\gamma^{-1}$$

for $\mathcal{F} = \gamma i\infty$, where $\mathcal{F}$ runs through the cusps of $\Gamma(N)^A \backslash (\mathbb{H} \times \mathbb{C}^n)$.
Then: The toroidal compactification

$$\overline{\mathcal{E}^{(n)}}^{tor} = \overline{\Gamma(N)^A \backslash (\mathbb{H} \times \mathbb{C}^n)}^{tor}_{\Sigma}$$

of the Kuga-Sato variety

$$\mathcal{E}^{(n)} = \Gamma(N)^A \backslash (\mathbb{H} \times \mathbb{C}^n)$$

of level N and rank n is a smooth projective variety.

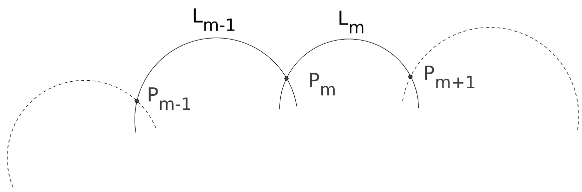
Taken altogether, the preceding sections illustrates how to get a compactification of the n-th fiber product $\mathcal{E}^{(n)}$ of the universal elliptic curve $\mathcal{E}$ over the modular curve $Y(N)$ from a suitable fan. We'll study the case $n = 1$ in further detail before proceeding to lattices of higher rank.

A fan for the case $n = 1$

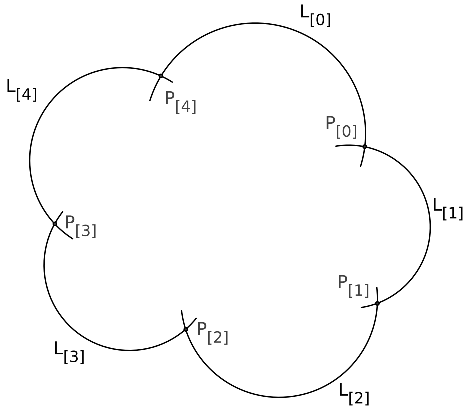
We want to take a closer look at a particular choice of Σ as in the preceding lemma for the case of $n = 1$. This is low-dimensional enough to actually draw some explanatory pictures. We follow the treatment in [AMRT10, Chapter 1, Section 4] quite closely.
In this case the group $\Gamma(N)_0^A$ is generated by a single element α acting as the translation $(\tau, z) \mapsto (\tau, z + \tau)$ on $\mathbb{C} \times \mathbb{C}$ and as $(q, x) \mapsto (q, q^N x)$ on $\mathbb{C}^* \times \mathbb{C}^*$ (remember the $1/N$ in the exponential map defining the isomorphism $\Delta^* \times (\mathbb{C}^*)^n \cong (\mathbb{H} \times \mathbb{C}^n) / \Gamma(N)_1^A$). On the level of the cocharacter lattices we have $N(\mathbb{C}^* \times \mathbb{C}^*) \cong \mathbb{Z}^2$ with the action of α as $(a, b) \mapsto (a, Na + b)$. We choose $\Sigma = \Sigma_{\mathrm{ell}}$ to be the fan containing the full-dimensional cones

$$\sigma_m = \mathbb{R}_+(1, m) + \mathbb{R}_+(1, m + 1)$$

for $m \in \mathbb{Z}$ as well as its faces. It is smooth and decomposes $\mathbb{R}_+ \times \mathbb{R}$, moreover we see that α maps σ_m to σ_{m+N}, so there are only finitely many cones modulo $\{\alpha^l | l \in \mathbb{Z}\}$, hence this is a fan Σ satisfying the conditions of proposition 3.3.3 and defines a smooth compactification $\overline{\mathcal{E}}_\Sigma^{\mathrm{tor}}$ of $\mathcal{E}$.
This compactification adds an N-gon of rational curve over every cusp: We note that $X_\Sigma \setminus (\mathbb{C}^*)^2$ consists of an infinite set of complex lines L_m (corresponding to the cones $\tau_m = \mathbb{R}_+(1, m)$). These intersect each other in a point P_m (corresponding to the cone $\sigma_m = \mathbb{R}_+(1, m) + \mathbb{R}_+(1, m+1)$) if and only if τ_m and τ_n span a σ_l, that is, if $|n - m| = 1$, so we can think of this as an infinite chain.



The action of α identifies every segment with its N-th predecessor and its N-th successor and hence yields a closed N-gon of rational curves. This is the object attached at the boundary in the partial compactification with respect to $i\infty$ and can be pictured as follows:



The compactification $\overline{\mathcal{E}}_{\Sigma}^{\mathrm{tor}}$ defined by this choice of Σ will be denoted simply by $\overline{\mathcal{E}}$ in the future.
There is a close connection between elliptic modular forms and elliptic curves, since $\Gamma(N)\backslash\mathbb{H}$ can be considered as the moduli space of elliptic curves with level-N-structure; simultaneously, this is the domain of elliptic modular forms as objects on locally symmetric spaces. This can be related to the universal elliptic curve $\Gamma(N)^A\backslash(\mathbb{H}\times\mathbb{C}) = \mathcal{E}$ and its compactification $\overline{\mathcal{E}}$ with corresponding projection $\pi : \overline{\mathcal{E}} \to \overline{Y(N)}$ we just considered. With the appropriate generalization for general $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, there is the following general result:

Lemma 3.3.5 ([Lan12a, Section 2] as well as [DR73, Section VII.4])**.** *Modular forms of weight 2 and level $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ can be identified with the sections of*

$$\omega = \pi_* \left(\Omega_{\overline{\mathcal{E}}/\overline{Y(\Gamma)}}\right),$$

the push-forward of the relative cotangent bundle of the family $\pi : \overline{\mathcal{E}} \to \overline{Y(\Gamma)}$. Modular forms of level Γ and weight $2k$ correspond to sections of the k-th tensor power of ω, that is

$$M_{2k}(\Gamma) = H^0\left(\overline{Y(\Gamma)}, \omega^{\otimes k}\right).$$

The bundle ω is sometimes called the *Hodge bundle* or the *bundle of modular forms*. By abuse of notation we will sometimes denote $\omega^{\otimes k}$ by $M_k(\Gamma)$.
In the next section we will give an example of a natural toroidal compactification for the case of rank $n > 1$.

Fans for the case $n > 1$

We notice that the fan $\Sigma = \Sigma_{\mathrm{ell}}$ used in the preceding section is exactly the one of example 2.2.3 and, due to the (in this case) local nature of the construction, we still only need to work over a fixed cusp $i\infty$ and translate everything afterwards. We already generalized this fan to higher dimensions by the fiber power construction of example 2.2.4:

Consider again the n-th fiber power Σ_{ell}^n of the fan Σ_{ell} over the trivial fan $\{\{0\}, \mathbb{R}\}$. This is a rational polyhedral cone decomposition of $\mathbb{R}_+ \times \mathbb{R}^n$ and has only finitely many equivalence classes under the action of $\Gamma(N)_0^A$, so we can use it to construct a compactification of the n-fold fiber product $\mathcal{E}^{(n)}$ of $\mathcal{E}$ over $Y(N)$.
Note that any full-dimensional cone in Σ is generated by 2^n edges which hence do not constitute a basis of $\mathbb{Z}^{n+1}$ for $n > 1$, so this fan is singular.
Applying the construction as described before in lemma 3.3.4 for the fan Σ_{ell}^n, we get a (non-smooth) compactification of $\mathcal{E}^{(n)}$ denoted by $\overline{\mathcal{E}^{(n)}}_{\text{ell}}$.
We describe the fibers of $\pi_n : \overline{\mathcal{E}^{(n)}}_{\text{ell}} \to \overline{Y(N)}$: For any point $p \in Y(N)$, the fiber $\pi^{-1}(p)$ is isomorphic to the n-fold product of the elliptic curve corresponding to $p \in Y(N)$ since this is the definition of the n-fold fiber product of the universal elliptic curve over $Y(N)$.
For $p \in \overline{Y(N)} \setminus Y(N)$, things are more complex: The fiber is the n-fold product of rational N-gons as one can see in the following. The toric variety in every partial compactification of $\mathcal{E}^{(n)}$ is just the n-th fiber power of the toric variety in the partial compactification of $\mathcal{E}$, where the fiber over a given boundary point $p \in \overline{Y(N)} \setminus Y(N)$ is just an infinite chain of rational curves; hence, the fiber over $p \in \overline{Y(N)} \setminus Y(N)$ in the partial compactification of $\mathcal{E}^{(n)}$ is the direct product of n of these infinite chains. The group $\Gamma(N)_0^A$ acts in each factor of this product by identifying each rational curve with its N-th successor and N-th predecessor, yielding a direct product of n rational N-gons as claimed. As there are no adjoint cusps in this case, the gluing procedure works locally, so the preceding description is valid for the toroidal compactification itself, not only for the partial compactifications.
These fibers can be decomposed into s-dimensional strata as follows: To get an s-dimensional stratum, select s of the n direct factors and choose one of the N rational curves in it; in the remaining $n - s$ direct factors choose one of the N intersection points of the rational curves constituting the N-gon. The direct product of these is a s-dimensional stratum of the fiber and any stratum is of this form. Easy combinatorics yield that there are exactly

$$\binom{n}{s} \cdot (N^s \cdot N^{n-s}) = \binom{n}{s} N^n$$

s-dimensional strata of this form.
There is another obvious way of compactifying $\mathcal{E}^{(n)}$: One can form the n-th scheme-theoretic fiber power $\overline{\mathcal{E}}^{(n)}$ of the compactification $\overline{\mathcal{E}}_{\text{ell}}$ constructed in section 3.3 over $\overline{Y(N)}$. This is done for example by Gordon in [Gor93, Section 1.b)].
An useful observation is the following, which can be easily verified by comparing the corresponding fibers described as before and as in [Gor93].

Lemma 3.3.6. *The compactifications*

$$\overline{\mathcal{E}^{(n)}}_{ell} \text{ and } \overline{\mathcal{E}}^{(n)}$$

coincide.

In other words: The diagram (of construction techniques, not actual morphisms between these spaces)

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & \overline{\mathcal{E}} \arrow[d] \\ \mathcal{E}^{(n)} \arrow[r] & Z \end{tikzcd}$$

with $Z = \overline{\mathcal{E}^{(n)}}_{\text{ell}} = \overline{\mathcal{E}}_{\text{ell}}{}^{(n)}$ commutes; here, the horizontal arrows represent the construction of toroidal compactifications by Σ_{ell} resp. Σ^n_{ell}, while the vertical arrows represent taking the n-fold fiber power over $Y(N)$ resp. $\overline{Y(N)}$.
Choosing different cone decompositions of the same cone (thus having the same support) will yield different toroidal compactifications; nevertheless, these are quite closely related to each other as seen in remark 3.1.11 if one of the defining admissible families is a refinement of the other. Since there always exists a common refinement of cone decompositions with the same support, we can relate any two of them:

Proposition 3.3.7. *Let Σ be an admissible cone decomposition with support $|\Sigma| = \mathbb{R}_+ \times \mathbb{R}^n$ such that $\overline{X}^{tor}_{\Sigma}$ is smooth and projective. Then the common refinement $\Sigma_0 = \Sigma \wedge \Sigma^n_{ell}$ of Σ and Σ^n_{ell} is such that, in the diagram*

$$\begin{tikzcd} & \overline{X}^{tor}_{\Sigma_0} \arrow[dl, "f_1"] \arrow[dr, "f_2"'] & \\ \overline{X}^{tor}_{\Sigma} & & \overline{\mathcal{E}}_{ell}{}^{n} \end{tikzcd}$$

the maps f_1 and f_2 are birational and f_1 is proper. If for any cone of Σ^n_{ell} there are only finitely many cones of Σ_0 contained in it, the morphism f_2 is proper as well.

Proof. By lemma 2.1.19 and lemma 2.1.20 both morphisms are birational and f_1 is proper. For the properness f_2 one has to check locally (cf. [AMRT10, III, 7.6]) the further condition in lemma 2.1.20 which is exactly as in the statement. $\square$

This closes our description of the construction of toroidal compactifications. This theory allows us to build a plethora of compactifications of locally symmetric space. In the next chapter we will present an approach to utilize this for the computation of dimensions of spaces of automorphic forms.

4. Dimension formulas

As the title of this thesis suggests, we are interested in the dimensions of spaces of automorphic forms. While we will focus on certain special cases of this notion later on, this chapter presents a general framework for determining dimensions of these spaces.
The usual approach is to interpret automorphic forms as global sections of a suitable vector bundle $\mathcal{V}$ (which works by definition with the point of view of Deligne) and extract the dimension of the space in question from the *Euler characteristic* $\chi(\mathcal{V})$ by the use of vanishing theorems that ensure the triviality of the higher cohomology groups.
There is a well-developed theory for the computation of Euler characteristics with the *Hirzebruch–Riemann–Roch theorem* as the main tool. It expresses Euler characteristics of vector bundles on smooth projective compact varieties in terms of certain geometric data.
In the first section we give a short introduction to general intersection theory and the other algebro-geometric tools necessary for the approach just sketched; the second section will compute a general canonical dimension formula up to complicated error term. The better description of this latter term is the content of the third and last section.

4.1. General algebro-geometric tools

While many (or most) of the following theorems are applicable to schemes of general characteristic p, we will stick to the case of $p = 0$, or more precisely to working over the algebraically closed field $\mathbb{C}$ of complex numbers, so any scheme S comes with a structure morphism $S \to \mathbb{C}$.
We start with a short run-through of the parts of general intersection theory we will need.

Intersection theory

We will follow the classic treatment in [Ful98]. The central object in intersection theory is the *Chow ring* defined in the following way:

Definition 4.1.1. Let X be a non-singular algebraic variety of dimension n. A *k-cycle* is a finite formal sum

$$\sum_i n_i V_i$$

of k-dimensional subvarieties V_i of X and integral weights $n_i \in \mathbb{Z}$. By $Z_k X$ we denote the free abelian group of codimension k-cycles on X. Dually, we can consider *codimension*

k*-cycle*; these are elements of $Z^k X = Z_{n-k} X$. Let Y be a $k+1$-dimensional subvariety of X and let f be a non-zero rational function on Y. This defines a k-cycle on X by

$$\operatorname{div}(f) = \sum \operatorname{ord}_Z(f) Z$$

where the sum runs over the dimension k-subvarieties Z of Y and $\operatorname{ord}_Z(f)$ is the *order of vanishing* of f along Z. If a k-cycle α can be written as

$$\alpha = \sum \operatorname{div}(f_i)$$

for f_i finitely many non-zero rational functions on $k+1$-dimensional subvarieties of X, it is called *rationally equivalent to zero*, written as

$$\alpha \sim 0.$$

We define *rational equivalence* of k-cycles $\alpha, \beta \in Z_k X$ by

$$a \sim \beta \text{ if and only if } \alpha - \beta \sim 0.$$

The quotient $A^k X = Z^k X / \sim$ by this relation is the *group of codimension* k*-cycles modulo rational equivalence.* The equivalence class of a cycle $V \in Z_k X$ will be denoted by $[V] \in A_k X$. The direct sum

$$\bigoplus_{i=1}^{n} A_i X$$

is the *Chow group of* X which we will denote by $\mathrm{CH}(X)$. The k-th graded part is then denoted by $\mathrm{CH}^k(X)$.

In particular, the construction shows that any relation of the form $\lambda_1 V_1 \sim_V \lambda_2 V_2$ for a subvariety V of X holds also true in X as $\lambda_1 V_1 \sim_X \lambda_2 V_2$; however it need not be true that $[V_1]_X \neq_X 0$ even if $[V_1]_V \neq_V 0$.

It is possible to associate cycles to arbitrary subschemes: For Y any closed subscheme of X write $Y_1, \ldots, Y_t$ for the irreducible components of the reduced subscheme Y_{red}. The fundamental cycle $[Y]$ of Y is

$$[Y] = \sum_i l_i [Y_i]$$

with l_i the *multiplicity* (length of $\mathcal{O}_{Y, Y_i}$) of Y_i in Y.

The Chow group is equipped with a (grading-respecting) ring structure by the *intersection product*: Let Y, Z be properly intersecting subvarieties, that is

$$\operatorname{codim}(Y \cap Z) = \operatorname{codim}(Y) + \operatorname{codim}(Z)$$

and denote by W_i the irreducible components of $Y \cap Z$ (with the scheme-theoretic intersection, i.e. $Y \cap Z = Y \times_X Z$, the fiber product of schemes). The intersection product is given by

$$[Y] \cdot [Z] = \sum_j i(Y, Z; W_j)[W_j]$$

and $i(Y, Z; W_j)$ is the *intersection multiplicity* of Y and Z at W_j; analogously we define proper intersection for general cycle classes. For Y, Z with non-proper intersection the famous *Chow's moving lemma* asserts that there are Y', Z' rationally equivalent to Y resp. Z with proper intersection. For the exact definition of the intersection multiplicity see [Ful98, Chapter 8].

Definition 4.1.2. The group $\mathrm{CH}(X)$ with the intersection product and unit $[X]$ is called the *Chow ring* of X. On the graded parts of $\mathrm{CH}(X)$ it induces maps

$$\mathrm{CH}^k(X) \times \mathrm{CH}^l(X) \to \mathrm{CH}^{k+l}(X).$$

This ring structure satisfies some very natural axioms and is the unique one with this property. It is not at all clear that all of this is indeed well-defined; the interested reader is invited to consult the corresponding chapters of [Ful98].
Morphisms between schemes induce morphisms between the corresponding Chow rings:

Lemma 4.1.3 ([Ful98, Chapter 1])**.** *Let X, Y be schemes and let $f : X \to Y$ be a morphism.*

i) If f is proper, there is a group homomorphism $p_* : \mathrm{CH}(X) \to \mathrm{CH}(Y)$ called the push-forward map *that respects the gradings. On cycles classes $[A]$ for a subvariety $A \subseteq X$ it is given by*

$$f([A]) = \begin{cases} 0 & \textit{if } \dim(f(A)) < \dim(A) \\ n[f(A)] & \textit{if } \dim(f(A)) = \dim(A) \textit{ and } f|_A \textit{ is a map of degree } n \end{cases}$$

ii) There is a group homomorphism $f^* : \mathrm{CH}(Y) \to \mathrm{CH}(X)$ called the pullback*. If f is a morphism of smooth quasi-projective varieties and the subvariety $B \subseteq Y$ is smooth as well with*

$$\mathrm{codim}_X(f^{-1}(B)) = \mathrm{codim}_Y(B),$$

the pullback is given by

$$f^*([B]) = [f^{-1}(B)],$$

with $f^{-1}(B)$ denoting the scheme-theoretic inverse image of B under f.

Moreover, for $f : X \to Y$ and $g : Y \to Z$ morphisms we have

$$(g \circ f)^* = f^* \circ g^*.$$

If f, g are proper, we get

$$(g \circ f)_* = g_* \circ f_*.$$

There is an important fact relating these two maps: the *projection formula.*

Lemma 4.1.4. *Let $f : X \to Y$ be a proper morphism. For $\alpha \in \mathrm{CH}_k(X)$ and $\beta \in \mathrm{CH}^l(Y)$ we have*

$$f_*(\alpha) \cdot_Y \beta = f_*\left(\alpha \cdot_X f^*(\beta)\right) \in \mathrm{CH}_{l-k}(Y).$$

The following easy corollary shows the locality of intersection products.

Corollary 4.1.5. *If $f : X \to Y$ is a closed immersion of quasi-projective varieties, then, for any smooth subvariety $B \subseteq Y$ with $\operatorname{codim}_X(f^{-1}(B)) = \operatorname{codim}_Y(B)$, we have*

$$[X] \cdot_Y [B] = f_*([B \cap X]_X).$$

Proof. Simple computation using the properties of f_*, f^* just described:

$$[X] \cdot_Y [B] = f_*([X]_X) \cdot_Y [B] = f_*\left([X] \cdot_X f^*([B])\right) = f_*(f^*([B])) = f_*([B \cap X]_X).$$

$\square$

A word of caution is needed here: The scheme $B \cap X$ is the *scheme-theoretic intersection* of B and X which may differ as a scheme from the closed reduced subscheme corresponding to the set-theoretic intersection of the variety underlying B and X, even if the intersection is proper. In that case, the difference is encoded in the geometric multiplicity of the components of $B \cap X$. To make the point even more concrete: For irreducible scheme-theoretic intersection $B \cap X$ we may have

$$[(B \cap X)_{\mathrm{Sch}}] = m\,[(B \cap X)_{\mathrm{Set}}]$$

with $m > 1$. We adopt the usual convention to mean by $B \cap X$ the scheme-theoretic intersection and keep in mind, that there may still a multiplicity lurking around if we are treating these scheme-theoretic intersections as being the same as the set-theoretic intersections.
Since inverse image schemes are also defined by the fiber product of schemes, these comments apply accordingly to the difference between the scheme-theoretic inverse images and their set-theoretic counterparts. As for intersections, we mean by $f^{-1}(B)$ the scheme-theoretic inverse image, so $\left[(f^{-1}(B))_{\mathrm{Sch}}\right]$ may differ by multiplicities from the cycle of the reduced subscheme corresponding to the set-theoretic inverse image.

Remark 4.1.6. A proper complex scheme X comes with a *degree morphism*

$$\deg : \mathrm{CH}_0(X) \to \mathbb{Z}$$

defined as the push-forward $p_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(\mathbb{C}) \cong \mathbb{Z}$ of the structure morphism $p : X \to \mathbb{C}$ on 0-cycles and trivially zero elsewhere.
By the functoriality of push-forwards we see

$$\deg(f_*\alpha) = ({p_X}_* \circ f_*)(\alpha) = (p_X \circ f)_*(\alpha) = {p_Y}_*(\alpha) = \deg(\alpha)$$

for any morphism $f : X \to Y$ of proper complex schemes.

This map allows us to associate integers to 0-cycles. The standard application of this is the concept of *intersection number*: Given a collection $\alpha_1, \ldots, \alpha_l \in \mathrm{CH}(X)$ of cycle classes with $\alpha_i \in \mathrm{CH}^{k_i}(X)$ such that $k_1 + \ldots + k_l = \dim X$, the intersection product $\alpha_1 \cdot \ldots \cdot \alpha_l$ is a 0-cycle and the associated integer

$$\deg(\alpha_1 \cdot \ldots \cdot \alpha_l)$$

is called the *intersection number* of these cycle classes (with each other).

Remark 4.1.7. In some cases, the actual computation of intersection numbers can be carried out in the Chow ring of one of the factors: Let $X_1, \ldots, X_l$ be smooth subschemes of a smooth proper scheme Y that satisfy pairwise the condition in *ii*) of lemma 4.1.3 and denote the inclusion morphisms by $\iota_i : X_i \to Y$. Then repeated use of the projection formula in the form of corollary 4.1.5 shows that

$$\begin{aligned}
[X_1] \cdot_Y [X_2] \cdot_Y [X_3] \cdot_Y \ldots \cdot_Y [X_l] &= ([X_1] \cdot_Y [X_2]) \cdot_Y [X_3] \cdot_Y \ldots \cdot_Y [X_l] \\
&= (\iota_{1*}([X_1 \cap X_2])) \cdot_Y [X_3] \cdot_Y \ldots \cdot_Y [X_l] \\
&= (\iota_{1*}([X_1 \cap X_2]) \cdot_{X_1} \iota_1{}^*([X_1 \cap X_3])) \cdot_Y \ldots \cdot_Y [X_l] \\
&= (\iota_{1*}([X_1 \cap X_2] \cdot_{X_1} [X_1 \cap X_3]) \cdot_Y \ldots \cdot_Y [X_l] \\
&= \cdots \\
&= \iota_{1*}\left([X_1 \cap X_2] \cdot_{X_1} [X_1 \cap X_3] \cdot_{X_1} \ldots \cdot_{X_1} [X_1 \cap X_l]\right),
\end{aligned}$$

but the push-forward morphism i_{1*} is irrelevant for the computation of the intersection number, so we get

$$\deg\left([X_1] \cdot_Y [X_2] \cdot_Y \ldots \cdot_Y [X_l]\right) = \deg\left([X_1 \cap X_2] \cdot_{X_1} \ldots \cdot_{X_1} [X_1 \cap X_l]\right).$$

This is useful as it allows induction/reduction arguments via the dimension of the scheme whose Chow ring the computations takes place in. We will often use this kind of reasoning during computations of intersection numbers without mentioning this chain of arguments explicitly again.

Remark 4.1.8. A general word of caution: We will use implicitly the categorical equivalence of the category of integral schemes of finite type over $\operatorname{Spec}\mathbb{C}$ with the category irreducible algebraic $\mathbb{C}$-varieties, by working with closed $\mathbb{C}$-points while still talking about the corresponding schemes. Moreover, we will identify the appearing complex algebraic varieties with their corresponding analytic spaces whenever this makes sense, and thereby seemingly applying analytic techniques to varieties and geometric techniques to manifolds. This is justified by the famous GAGA-principle (*Géometrie Algébrique et Géométrie Analytique*) by Serre.

Sheaves and characteristic classes

Later on we will need some more general tools from algebraic geometry which we will list here for convenience.

While we are interested in the dimension of the space of global sections of a certain sheaf, this is a quantity not really well-behaved (i.e. invariant) under algebraic and geometric operations. A more useful and robust quantity (which is, for example, invariant under birational transformation) is the Euler characteristic of a sheaf.

Definition 4.1.9. Let X be a projective scheme over a field k, and let $\mathcal{F}$ be a coherent sheaf on X. The *Euler characteristic* $\chi(\mathcal{F})$ of $\mathcal{F}$ is the alternating sum of the dimensions of the i-th sheaf cohomology

$$\chi(\mathcal{F}) = \sum_{i=-\infty}^{\infty} (-1)^i \dim H^i(X, \mathcal{F}).$$

Remark 4.1.10. This is a central tool in the computation of the space of global sections of a sheaf. Our approach to dimension formulas is a classical example for the techniques used there. Note that Euler characteristics are additive on short exact sequences, so

$$\chi(\mathcal{F}_2) = \chi(\mathcal{F}_1) + \chi(\mathcal{F}_3)$$

for exact

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0.$$

We can associate so-called *Chern classes* to any locally free sheaf of rank r on a non-singular quasi-projective variety by the following construction:

Proposition 4.1.11. *There is a unique ring homomorphism $c : K_0(X) \to \mathrm{CH}(X)$ mapping the Grothendieck group $K_0(X)$ of X to the Chow ring $\mathrm{CH}(X)$ satisfying the following conditions:*
Let E be a locally free sheaf and decompose $c(E) = c_0(E)+c_1(E)+\dots$ into graded pieces. Then:

i) $c_0(E) = 1$

ii) $c_1(\mathcal{O}_X(D)) = [D]$ *for any invertible sheaf* $\mathcal{O}_X(D)$

iii) $c(E) = c(E')c(E'')$ *for any short exact sequence*

$$0 \to E' \to E \to E'' \to 0$$

iv) $c_i(E) = 0$ *for* $i > \mathrm{rank}(E)$

This allows us to define:

Definition 4.1.12. Let E be a locally free sheaf on X. The i-th graded part of the total Chern class $c(E) \in A^*X$ is called the *i-th Chern class of E*. The image $c(E)$ itself is called the *total Chern class of E*.

For the application in mind, there is the technical notion of *Chern characters* that behaves very well with respect to the standard operations on sheaves.

Definition 4.1.13. For a locally free sheaf L of rank 1 on X we define the *Chern character* to be the formal series

$$\mathrm{ch}(L) = \exp\left(c_1(L)\right) = \sum_{k=0}^{\infty} \frac{c_1(L)^k}{k!}.$$

For a direct sum $\mathcal{F} = L_1 \oplus \dots L_m$ of line bundles, this generalizes to

$$\mathrm{ch}(\mathcal{F}) = \sum_{i=1}^{m} \mathrm{ch}(L_i).$$

In terms of the Chern classes of $\mathcal{F}$, the Chern character $\operatorname{ch}(\mathcal{F})$ is

$$\operatorname{rank}(\mathcal{F}) + c_1(\mathcal{F}) + \frac{1}{2}\Big(c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})\Big) + \frac{1}{6}\Big(c_1(\mathcal{F})^3 - 3c_1(\mathcal{F})c_2(\mathcal{F}) + 3c_3(\mathcal{F})\Big) + \dots$$

By the *splitting principle* (cf. [BT82]), this last formula is valid even for vector bundles that are not a direct sum of line bundles by formal computations in a suitable enlarged ring.
Since Chern classes vanish for index higher than the rank of $\mathcal{F}$, the Chern character is actually a polynomial in the Chern classes.
The Chern character satisfies

- $\operatorname{ch}(L_1 \oplus L_2) = \operatorname{ch}(L_1) + \operatorname{ch}(L_2)$ and
- $\operatorname{ch}(L_1 \otimes L_2) = \operatorname{ch}(L_1)\operatorname{ch}(L_2)$.

We need to introduce one further notion:

Definition 4.1.14. Let

$$Q(x) = \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}B_i}{(2i)!}x^{2i} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots$$

be the formal power series expansion of $Q(x)$ where B_i is the i-th Bernoulli number. For a line bundle L, we define its *Todd class* to be

$$\operatorname{td}(L) = Q(c_1(L)) = 1 + \frac{1}{2}c_1(L) + \frac{1}{24}c_1(L)^2 - \frac{1}{720}c_1(L)^4 + \dots$$

For a direct sum $\mathcal{F} = L_1 \oplus \dots L_m$ of line bundles, we define

$$\operatorname{td}(\mathcal{F}) = \prod_{i=1}^{m} Q(c_1(L_i)).$$

By the splitting principle, this already defines the Todd class of an arbitrary vector bundle on X.

Again, this can be reformulated in the Chern classes of $\mathcal{F}$:

$$\operatorname{td}(\mathcal{F}) = 1 + \frac{1}{2}c_1(\mathcal{F}) + \frac{1}{12}\Big(c_1(\mathcal{F})^2 + c_2(\mathcal{F})\Big) + \frac{1}{24}c_1(\mathcal{F})c_2(\mathcal{F}) + \dots$$

With these definition we are able to state the Hirzebruch-Riemann-Roch theorem:

Theorem 4.1.15 ([Har77], Appendix A, Theorem 4.1; or originally [Hir95])**.** *Let $\mathcal{E}$ be a locally free sheaf of rank r on a smooth projective variety X of dimension n. The Euler characteristic of $\mathcal{E}$ is given*

$$\chi(X, \mathcal{E}) := \chi(\mathcal{E}) = \deg(\operatorname{ch}(\mathcal{E}).\operatorname{td}(\mathcal{T}_X))$$

with $\mathcal{T}_X$ the tangent sheaf of X.

This is to be understood in the following way: The product of the Chern character $\mathrm{ch}(\mathcal{E})$ and the Todd class $\mathrm{td}(\mathcal{T}_X)$ is a sum of intersection products of the Chern classes $c_i(\mathcal{E})$ and $c_j(\mathcal{T}_X)$ in various degrees; the degree morphism is trivial except on those products of combined degree $\dim X$. This is exactly an intersection number.
For the more analytical-minded reader: An (GAGA)-equivalent formulation of the requirements is that $\mathcal{E}$ is a holomorphic rank r vector bundle on a smooth compact complex manifold X.
An equivalent and widely used notation for the Hirzebruch–Riemann–Roch theorem is

$$\int_X \mathrm{ch}(\mathcal{E})\mathrm{Td}(X) := \deg(\mathrm{ch}(\mathcal{E}).\mathrm{td}(\mathcal{T}_X)),$$

interpreting the degree map as an integral on X and emphasizing the dependence of $\mathrm{td}(\mathcal{T})$ only on the geometry of X. We will use these notations interchangeably.
In the case of X being a complete non-singular algebraic curve and $\mathcal{E}$ being an invertible sheaf (i.e. X a compact Riemann surface and $\mathcal{E}$ a holomorphic line bundle), this reduces to the classical Riemann-Roch theorem by noting $\deg(c_1(\mathcal{T})_1 = 2 - 2g$:

Corollary 4.1.16. *For X a compact Riemann surface with tangent bundle $\mathcal{T}$ and $\mathcal{L}$ a holomorphic line bundle on X we have*

$$\begin{aligned} \chi(X,\mathcal{L}) &= \dim H^0(X,\mathcal{L}) - \dim H^1(X,\mathcal{L}) \\ &= c_1(\mathcal{L}) + \frac{1}{2}c_1(\mathcal{T}) \\ &= \deg(D) + 1 - g \end{aligned}$$

where D is the divisor corresponding to $\mathcal{L}$ and g is the genus of X.

The even more classical form

$$l(D) - l(K - D) = \deg(D) + 1 - g$$

follows by an application of Serre duality, see proposition 4.1.21.

Example 4.1.17. Specializing theorem 4.1.15 to the case of the trivial sheaf $\mathcal{O}_X$, one gets

$$\chi(\mathcal{O}_X) = \int_X \mathrm{ch}(\mathcal{O}_X)\mathrm{Td}(X) = \int_X \mathrm{Td}(X) = \deg_n \mathrm{Td}(X).$$

For a toric variety $X = X_\Sigma$, for example, the latter can be expressed in the closures $D_\rho = \overline{O(\rho)}$ of the orbits corresponding to rays ρ in Σ by the orbit-cone correspondence (cf. proposition 2.1.11) and [CLS11, Theorem 13.1.6] as follows:

$$\chi(\mathcal{O}_{\mathcal{X}}) = \deg_n \prod_{\rho \text{ ray}} \frac{[D_\rho]}{1 - e^{-[D_\rho]}}.$$

By multiplying and collecting Chern classes, one gets the following:

Corollary 4.1.18. *The Euler characteristic of a sheaf $\mathcal{F}$ can be expressed in the form*

$$\chi(\mathcal{F}) = \sum_{|\alpha|+|\beta|=n} c_{\alpha,\beta} c^{\alpha}(\mathcal{F}) c^{\beta}(\Omega^1_X)$$

where $c_{\alpha,\beta} \in \mathbb{Q}$ are coefficients only depending on the dimension $n = \dim X$ of X. We denote this polynomial in the Chern classes $c_i(\mathcal{F})$ and $c_j(\Omega^1_X)$ by Q_n or Q if the dimension is clear from context.

Following [Fio17] we can consider this polynomial abstractly. If we write $Q_n(\underline{x}; \underline{y})$ for $\underline{x} = (x_0, \ldots, x_n), \underline{y} = (y_0, \ldots, y_n)$ with $_i, y_i \in \mathrm{CH}_i(X)$ any cycles, we understand

$$Q_l(\underline{x}, \underline{y})$$

as the polynomial expression in x_i, y_i obtained by replacing $c_i(\mathcal{F})$ by x_i and $c_i(\Omega^1_X)$. We give some examples of the abstract polynomials:

Example 4.1.19. We have:

$$\begin{aligned}
Q_0(x_0) &= x_0 \\
Q_1(x_0, x_1; y_1) &= \frac{1}{2} x_0 y_1 + y_0 \\
Q_2(x_0, x_1, x_2; y_1, y_2) &= \frac{1}{12} x_0 \left(y_1^2 + y_2\right) + \frac{1}{2} x_1 y_1 \\
Q_3(x_0, x_1, x_2, x_3; y_1, y_2, y_3) &= \frac{1}{24} x_0 y_1 y_2 + \frac{1}{12} x_1 \left(y_1^2 + y_2\right) + \ldots
\end{aligned}$$

To make use of the concept of Euler characteristics, one usually uses vanishing theorems to force the collapse of the higher cohomologies, so the Euler characteristic actually equals the dimension of $H^0(X, \mathcal{F})$.

Theorem 4.1.20 ([Har77, Chapter III, Remark 7.15])**.** *Let X be a projective non-singular variety of dimension n over $\mathbb{C}$ and $\mathcal{L}$ an ample invertible sheaf on X, then*

i) $H^i(X, \mathcal{L} \otimes \omega_X) = 0$ *for* $i > 0$

ii) $H^i(X, \mathcal{L}^{-1}) = 0$ *for* $i < 0$

We will use the following corollary of the well-known *Serre duality*:

Proposition 4.1.21 ([Har77], Chapter III, Corollary 7.7)**.** *Let X be a non-singular projective Cohen-Macaulay scheme of equidimension n over an algebraically closed field k. For any locally free sheaf $\mathcal{F}$ on X there are natural isomorphisms*

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \check{\mathcal{F}} \otimes \omega_X).$$

The following bundle will be of particular importance later on.

Definition 4.1.22. Let Δ be a normal crossing divisor of a complex analytic variety X and denote the inclusion of $X^* = X \setminus \Delta$ by $j : X^* \to X$. Assume that Δ is locally given by the vanishing of $q_1 \cdot \ldots \cdot q_k$ with $q_1 \ldots, q_n$ the local coordinates.
We denote by $\Omega^1_X(\log \Delta)$ the locally free sub-$\mathcal{O}_X$-module of $j_*\Omega^1_{X^*}$ of elements of the form

$$\sum_{i=1}^{k} a_i(z)\frac{dz_i}{z_i} + \sum_{j=k+1}^{n} b_j(z)dz_j.$$

We call $\Omega^p_X(\log \Delta) = \bigwedge^p \Omega^1_X(\log \Delta)$ the *sheaf of differential p-forms with logarithmic poles along* Δ or *sheaf of* Δ*-logarithmic p-forms.* If the divisor Δ is clear from the context, we will often use the notation $\Omega^p_X(\log)$ (cf. [EV92]).

For $p = 1$ we often speak of the *logarithmic cotangent bundle.*
In the case of toric varieties, this bundle is trivial.

Example 4.1.23 ([Tsu80, Theorem 2.4])**.** Let X_Σ be a smooth compact toric variety with torus T and denote by Δ its boundary divisor $X_\Sigma \setminus T$. Then $\Omega^1_X(\log \Delta)$ is trivial and all of its higher Chern classes are zero.

In general, this is a non-trivial and complex object. This bundle fits into the following short exact sequence:

Lemma 4.1.24. *In the situation of definition 4.1.22 we have the following exact sequence*

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log \Delta) \longrightarrow \bigoplus_j (i_j)_* \mathcal{O}_{D_j} \longrightarrow 0 \ ,$$

where we assume that $\Delta = \{D_i | \, i \in I\}$ *and* $i_j : D_j \hookrightarrow X$ *is the inclusion. In particular we get*

$$c(\Omega^1_{\overline{X}}) = c(\Omega^1_{\overline{X}}(\log)) \prod_j (1 - [D_j]).$$

Auxiliary results and techniques

For the sake of completeness we include the following well-known and general notions and results. The first few of these serve mainly to introduce the necessary notations.

Definition 4.1.25. Denote by $\mathrm{Sh}(Z)$ the category of sheaves of abelian groups on Z. Any continuous mapping $f : X \to Y$ of topological spaces gives rise to the *direct image functor*

$$f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$$

by sending a sheaf $\mathcal{F}$ to the sheaf $f_*(\mathcal{F})$ defined by

$$f_*(\mathcal{F}(U)) := \mathcal{F}\left(f^{-1}(U)\right)$$

for any open $U \subseteq Y$. This functor is left-exact but, in general, not right exact. Its right *derived functors* are called *higher direct images*

$$R^i f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$$

for $i \geq 0$ and for a sheaf $\mathcal{F}$ on X, the i-th higher direct image $R^i f_*(\mathcal{F})$ is the sheaf associated to the presheaf given on open $U \subseteq Y$ by

$$U \mapsto H^i\left(f^{-1}(U), \mathcal{F}|_U\right).$$

The higher direct images allow to push forward the computation of an Euler characteristic:

Proposition 4.1.26 ([Sta21, Lemma 0BEK])**.** *Let k be a field and let $f : X \to Y$ be a morphism of proper schemes over k. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have*

$$\chi(X, \mathcal{F}) = \sum_i (-1)^i \chi(Y, R^i f_* \mathcal{F}).$$

The following result, usually called *projection formula*, will also be useful later on.

Proposition 4.1.27 ([Sta21, Tag 01E8])**.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and $\mathcal{E}$ be an $\mathcal{O}_Y$-module which is finite locally free on Y. Then, for every $q \geq 0$, there is an isomorphism*

$$\mathcal{E} \otimes_{\mathcal{O}_Y} R^q f_* \mathcal{F} \cong R^q f_* \left(f^* \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}\right).$$

The next result is a generalization of the well-known Künneth formula and is stated quite generally in the language of derived algebraic geometry; we will soon reduce it to a more tangible form, so we will not introduce all of the notation used there. One notion we will introduce is the *Tor-independence*:

Definition 4.1.28. Let S be a scheme and X, Y schemes over S. The schemes X and Y are called *Tor-independent* if for every $x \in X$ and $y \in Y$ with identical image $s \in S$ the rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ satisfy

$$\mathrm{Tor}_p^{\mathcal{O}_{S,s}}\left(\mathcal{O}_{X,y}, \mathcal{O}_{Y,y}\right) = 0$$

for all $p > 0$.

The most important special case for our purposes is if either X or Y is flat over S: Then X, Y are Tor-independent by a standard result in commutative algebra.
The general Künneth theorem is:

Lemma 4.1.29 ([Sta21, Tag 0FLN])**.** *Let S be a scheme and X, Y schemes over S. Denote by $D_{QCoh}(\mathcal{O}_X)$ the derived category of quasi-coherent $\mathcal{O}_X$-modules and let K, M in $D_{QCoh}(\mathcal{O}_X)$ resp. $D_{QCoh}(\mathcal{O}_Y)$. Let*

$$\begin{tikzcd}
& X \times_S Y \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "f"] & \\
X \arrow[dr, "a"'] & & Y \arrow[dl, "b"] \\
& S &
\end{tikzcd}$$

be a Cartesian square of S-schemes. If the morphisms a, b are quasi-compact and X, Y are Tor-independent, the canonical morphism

$$Ra_*K \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*M \longrightarrow Rf_*(Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} Lq^*M)$$

is an isomorphism.

We will use this result only in a very special case:

Lemma 4.1.30. *Let X, Y be quasi-compact, separated and flat over a separated, quasi-compact and smooth S and let $\mathcal{F}$, $\mathcal{G}$ be coherent sheaves on X resp. Y. Then:*

$$R^n f_*\left(p^*(\mathcal{F}) \otimes_{X\times_S Y} q^*(\mathcal{G})\right) \cong \bigoplus_{i=0}^{n} \left(R^i a_*\left(\mathcal{F}\right) \otimes_{\mathcal{O}_S} R^{n-i} b_*\left(\mathcal{G}\right)\right).$$

Proof. This is an application of lemma 4.1.29: By flatness of the maps, the derived pullbacks and derived tensor products are just the usual pullback and tensor product. □

Higher direct images are a useful notion when working with singular spaces. Their vanishing can be used to define a certain class of very mild singularities.

Definition 4.1.31. A normal scheme X of finite type has *rational singularities* if there exists a regular scheme Y and a proper birational map $f : Y \to X$ with

$$R^i f_*\left(\mathcal{O}_Y\right) = 0 \qquad \text{for all } i > 0.$$

If a variety X has rational singularities, it is possible to replace the higher direct images of the structure sheaf of X by the higher direct images of the structure sheaf of a variety Y properly birational to X.

Proposition 4.1.32. *Let S be a scheme. Let X, Y be S-schemes with rational singularities and let Z be a smooth S-scheme such that there are proper birational morphisms f_X, f_Y as in the commutative diagram*

$$\begin{tikzcd} & Z \arrow[dl, "f_X"'] \arrow[dr, "f_Y"] \arrow[dd, "\pi_Z"] & \\ X \arrow[dr, "\pi_X"'] & & Y \arrow[dl, "\pi_Y"] \\ & S & \end{tikzcd}.$$

Then:

$$R^i \pi_{X*}\left(\mathcal{O}_X\right) = R^i \pi_{Y*}\left(\mathcal{O}_Y\right).$$

This is certainly well-known. For lack of a reference we give a short proof:

Proof. Obviously it suffices to show

$$R^i\pi_{X*}(\mathcal{O}_X) = R^i\pi_{Z*}(\mathcal{O}_Z)$$

by symmetry in X and Y. The relative Leray spectral sequence (see [Sta21, Tag 0734]) states that

$$R^i\pi_{X*}\left(R^jf_{X*}(\mathcal{O}_Z)\right) \Rightarrow R^{i+j}\pi_{Z*}(\mathcal{O}_Z),$$

and, by X having rational singularities, we have

$$R^jf_{X*}(\mathcal{O}_Z) = 0$$

for all $j > 0$, so the spectral sequence degenerates at the 2-sheet and yields

$$R^i\pi_{X*}\left(R^jf_{X*}(\mathcal{O}_Z)\right) = R^{i+j}\pi_{Z*}(\mathcal{O}_Z)$$

for all $i, j \geq 0$ which gives

$$R^i\pi_{Z*}(\mathcal{O}_Z) = R^i\pi_{X*}\left(R^0f_{X*}(\mathcal{O}_Z)\right) = R^i\pi_{X*}(f_{X*}(\mathcal{O}_Z)) = R^i\pi_{X*}(\mathcal{O}_X)$$

in particular. $\square$

We should point out that toric singularities and regular points are rational singularities (cf. [Ful93, p.76], so this is a frequently appearing concept.
We finish by a relative version of the already stated Serre duality:

Lemma 4.1.33 ([Kle80])**.** *Let X, Y be locally Noetherian schemes and $\pi : X \to Y$ a flat projective morphism of relative dimension n with geometric fibers being Cohen-Macaulay. Let $\mathcal{F}$ be a coherent sheaf on X. Then: The canonical dualizing sheaf $\omega_{X/Y}$ and the trace map $R^n\pi_*\omega_{X/Y} \to \mathcal{O}_Y$ give a perfect pairing*

$$R^i\pi_*\mathcal{F} \times R^{n-i}\pi_*(\mathcal{F}^\vee \otimes \omega_{X/Y}) \to R^n\pi_*\omega_{X/Y} \to \mathcal{O}_Y.$$

In other words:

$$R^i\pi_*\mathcal{F} \cong \left(R^{n-i}\pi_*\left(\mathcal{F}^\vee \otimes \omega_{X/Y}\right)\right)^\vee$$

for all $i \geq 0$.

This finishes our collection of general tools from algebraic geometry and lends rigor to the strategy for the computation of dimension formulas as in the introduction of this chapter. We will explain it in a rigorous way in the next section.

4.2. Computation of dimension formulas

To compute dimensions of spaces of automorphic forms of given weight k and level Γ on a symmetric space $\mathcal{D}$, in general one proceeds as follows:

- Realize weight k automorphic forms as global sections of a bundle $\mathcal{F}^{\otimes k}$ on the locally symmetric space $X = \Gamma\backslash\mathcal{D}$, so
$$\dim \mathcal{M}_k(\Gamma) = \dim H^0\left(X, \mathcal{F}^{\otimes k}\right).$$

- Use a vanishing theorem to see that $H^i\left(X, \overline{F}^{\otimes k}\right) = 0$ for $i > 0$ and suitably large weight $k \gg 0$, so
$$\dim \mathcal{M}_k(\Gamma) = \dim H^0\left(X, \mathcal{F}^{\otimes k}\right) = \sum_{i=0}^{\infty}(-1)^i \dim H^i\left(X, \mathcal{F}^{\otimes k}\right) = \chi\left(\mathcal{F}^{\otimes k}\right).$$

- If X is smooth and projective: Use the Hirzebruch–Riemann–Roch theorem to express this Euler characteristic as an intersection product of Chern classes of $\mathcal{F}^{\otimes k}$ and the tangent sheaf $\mathcal{T}_X$ of X.

- Compute the appearing intersection numbers.

Of course, this approach works only under the condition of the compactness of X, which is unfortunately often not fulfilled for naturally appearing locally symmetric spaces, namely those of non-compact type.
A remedy for this is given by smooth projective toroidal compactifications of smooth X.

Hirzebruch-Mumford proportionality and Mumford's corollary

To apply the strategy in the preceding section for a non-compact locally symmetric space $X = \Gamma\backslash\mathcal{D}$ one would like to compactify X to a smooth projective $\overline{X}$ and extend the bundle of automorphic forms to the compactification, such that the extended bundle is still closely related to the original bundle of automorphic forms.
All of this is achieved by the following theorem due to Mumford for toroidal compactifications:
For a bundle on the symmetric space $\mathcal{D}$ it constructs related bundles on the compact dual $\check{\mathcal{D}}$, the locally symmetric space X and its toroidal compactification $\overline{X}$. Moreover, it computes the Chern classes of the bundle of $\overline{X}$ in terms of the Chern classes of the bundle on the compact dual $\check{\mathcal{D}}$.

Theorem 4.2.1 ([Mum77, Theorem 3.2])**.** *Let $\mathcal{D}$ be a bounded symmetric domain of dimension n and Γ a discrete torsion-free co-compact subgroup of automorphisms of $\mathcal{D}$. Denote the compact dual of $\mathcal{D}$ by $\check{\mathcal{D}}$. Denote by $\overline{X} = \overline{X}^{tor}_{\Sigma}$ a smooth toroidal compactification of $X = \Gamma\backslash\mathcal{D}$.*
Let E_0 a G-equivariant vector bundle of rank r on $\mathcal{D}$.
Then there exist canonical vector bundles $\check{E}_0$ on $\check{\mathcal{D}}$, E on X and $\overline{E}$ on $\overline{X}$ related to E_0 as follows:

$$\begin{tikzcd}
\check{E}_0 \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \overline{E} \arrow[d] \\
\check{\mathcal{D}} & \mathcal{D} \arrow[l, hook'] \arrow[r] & X \arrow[r] & \overline{X}
\end{tikzcd},$$

the first upper arrow being the restriction, the second the descent by Γ *and the third given by extension.*
Moreover: There exists a constant $\mathrm{Vol}_{\mathrm{HM}}(\Gamma)$ *called* Hirzebruch-Mumford volume of X *such that for all multi-indices* $\alpha = (\alpha_1, \ldots, \alpha_r)$ *of weight* $\sum_{i=1}^r \alpha_i = n$ *the following equality holds for the Chern numbers of* $\overline{E}$ *and* $\check{E}_0$*:*

$$c^\alpha \left(\overline{E}\right) = (-1)^n \mathrm{Vol}_{\mathrm{HM}}(\Gamma) c^\alpha \left(\check{E}_0\right)$$

To apply this to the computation of dimension formulas for automorphic forms, we need to relate the bundles of automorphic forms to the bundles appearing in the preceding theorem 4.2.1.
Mumford goes on to describe the bundles $\overline{E}$ in some special cases which will be useful for this.
We remember the notion of the sheaf $\Omega^p_X(\log \Delta)$ of Δ-logarithmic p-forms. Then:

Lemma 4.2.2 ([Mum77, Proposition 3.4 a)])**.** *If* $E_0 = \Omega^1_{\mathcal{D}}$ *in theorem 4.2.1 is the cotangent bundle, the bundle* $\overline{E}$ *on* $\overline{X}$ *is given by*

$$\overline{E} = \Omega^1_{\overline{X}}(\log \Delta)$$

with $\Delta = \overline{X} \setminus X$ *the compactification divisor.*

In the following we will always have $\Delta = \overline{X} \setminus X$, so we can suppress it from the notation and write simply $\Omega^1_{\overline{X}}(\log)$ for $\Omega^1_{\overline{X}}(\log \Delta)$.
Similarly we can describe $\overline{E}$ for the canonical line bundle $E_0 = \Omega^n_{\mathcal{D}}$.

Lemma 4.2.3. *If* $E_0 = \Omega^n_D$ *in theorem 4.2.1 is the canonical line bundle, the bundle* $\overline{E}$ *on* $\overline{X}$ *is*

$$\overline{E} = \pi^*(\mathcal{O}_{\overline{X}^{BB}}(1))$$

for $\pi : \overline{X} \to \overline{X}^{BB}$ *the canonical morphism of the toroidal compactification* $\overline{X}$ *of* X *to its Baily-Borel compactification and* $\mathcal{O}_{\overline{X}^{BB}}(1)$ *is the ample line bundle of weight* 1 *modular forms on* $\overline{X}^{BB}$*.*

This is useful for computing dimension formulas for automorphic forms by the following result which accomplishes exactly the extension of ordinary automorphic forms on X to global sections of a bundle on the toroidal compactification.

Lemma 4.2.4 ([Mum77, Proposition 3.3])**.** *Let* $E_0 = \Omega^n_{\mathcal{D}}$ *be the canonical bundle and* E *resp.* $\overline{E} = \Omega^n_{\overline{X}}(\log \Delta)$ *the corresponding bundles on* $X = \Gamma \backslash \mathcal{D}$ *and* $\overline{X}$*.*
Then: The vector space $\mathcal{S}_k(\Gamma)$ *of automorphic cusp forms of weight* k *in the sense of section 1.2 with trivial representation* ρ *is isomorphic to the space of global sections of*

$$\Omega^n_{\overline{X}}(\log \Delta)^{\otimes(k-1)} \otimes \Omega^n_{\overline{X}}.$$

Finally, Mumford himself applies the previously sketched strategy to determine the dimension of spaces of automorphic forms and gets the following remarkable result:

Theorem 4.2.5 ([Mum77, Corollary 3.5])**.** *Fix a smooth projective toroidal compactification $\overline{X}$ of the locally symmetric space $X = \Gamma\backslash\mathcal{D}$.*
For $\mathcal{S}_k(\Gamma)$ the space of cusp forms of weight k and level Γ on the Hermitian symmetric space $\mathcal{D}$ in the sense of section 1.2 and $k \geq 2$ we have

$$\dim(\mathcal{S}_k(\Gamma)) = \mathrm{Vol}_{\mathrm{HM}}(\Gamma)\mathcal{P}(k-1) + E(k)$$

with a polynomial $E(k)$ of degree

$$\deg(E) \leq \mathbb{P}\dim\left(\overline{X}^{BB} \setminus X\right)$$

and the Hilbert polynomial $\mathcal{P}(k) = \chi\left(\left(\Omega^n_{\check{D}}\right)^{-k}\right)$ of the compact dual $\check{D}$.

The polynomial $E(k)$ will henceforth be called *error term* of the dimension formula. For better understanding of the structure of the appearing objects, we reproduce the proof of this result in rather great detail:

Proof. We have

$$\dim \mathcal{S}_k(\Gamma) = \dim H^0\left(\overline{X}, \Omega^n_{\overline{X}}(\log)^{\otimes(k-1)} \otimes \Omega^n_{\overline{X}}\right)$$

by lemma 4.2.4. If $k \geq 2$ the higher cohomology groups of this bundle vanish by theorem 4.1.20, so

$$\begin{aligned}
\dim\left(\mathcal{S}_k(\Gamma)\right) &= \dim H^0\left(\overline{X}, \Omega^n_{\overline{X}}(\log)^{\otimes(k-1)} \otimes \Omega^n_{\overline{X}}\right) \\
&= \sum_{i\geq 0}(-1)^i \dim H^i\left(\overline{X}, \Omega^n_{\overline{X}}(\log)^{\otimes(k-1)} \otimes \Omega^n_{\overline{X}}\right) \\
&= \chi\left(\Omega^n_{\overline{X}}(\log)^{\otimes(k-1)} \otimes \Omega^n_{\overline{X}}\right).
\end{aligned}$$

With Serre duality (cf. proposition 4.1.21) this equals

$$\dim \mathcal{S}_k(\Gamma) = (-1)^n \chi\left(\Omega^n_{\overline{X}}(\log)^{\otimes(1-k)}\right).$$

The Hirzebruch-Riemann-Roch theorem implies the existence of a universal polynomial Q_n in the Chern classes of the cotangent bundle $\Omega^1_{\overline{X}}$ of $\overline{X}$ (cf. corollary 4.1.18) with

$$\dim \mathcal{S}_k(\Gamma) = (-1)^n Q_n\left((1-k)c_1\left(\Omega^n_{\overline{X}}(\log)\right); c_1\left(\Omega^1_{\overline{X}}\right), \ldots, c_n\left(\Omega^1_{\overline{X}}\right)\right);$$

this can be (trivially) written as

$$\dim \mathcal{S}_k(\Gamma) = (-1)^n Q_n\left((1-k)c_1\left(\Omega^n_{\overline{X}}(\log)\right); c_i\left(\Omega^1_{\overline{X}}(\log)\right)\right) + E(k)$$

with

$$\begin{aligned}
E(k) \;=\; & (-1)^n \Big[Q_n\left((1-k)c_1\left(\Omega^n_{\overline{X}}(\log)\right); c_1\left(\Omega^1_{\overline{X}}\right), \ldots, c_n\left(\Omega^1_{\overline{X}}\right)\right) \\
& - Q_n\left((1-k)c_1\left(\Omega^n_{\overline{X}}(\log)\right); c_1\left(\Omega^1_{\overline{X}}(\log)\right), \ldots, c_n\left(\Omega^n_{\overline{X}}(\log)\right)\right)\Big].
\end{aligned}$$

Note that the first Chern class of the cotangent bundle and the canonical bundle are equal, so

$$c_1\left(\Omega^n_{\overline{X}}(\log)\right) = c_1\left(\Omega^1_{\overline{X}}(\log)\right),$$

and we can write this as

$$\dim \mathcal{S}_k(\Gamma) = (-1)^n Q_n\left((1-k)c_1\left(\Omega^1_{\overline{X}}(\log)\right); c_i\left(\Omega^1_{\overline{X}}(\log)\right)\right) + E_{\overline{X}}(k).$$

By Hirzebruch-Mumford proportionality this is equal to

$$\begin{aligned}
\dim \mathcal{S}_k(\Gamma) &= \mathrm{Vol}_{\mathrm{HM}}(X) \cdot Q_n\left((1-k)c_1\left(\Omega^1_{\check{\mathcal{D}}}\right); c_1\left(\Omega^1_{\check{\mathcal{D}}}\right), \ldots, c_n\left(\Omega^1_{\check{\mathcal{D}}}\right)\right) + E(k) \\
&= \mathrm{Vol}_{\mathrm{HM}}(X) \cdot Q_n\left((1-k)c_1\left(\Omega^n_{\check{\mathcal{D}}}\right); c_1\left(\Omega^1_{\check{\mathcal{D}}}\right), \ldots, c_n\left(\Omega^1_{\check{\mathcal{D}}}\right)\right) + E(k) \\
&= \mathrm{Vol}_{\mathrm{HM}}(X) \cdot \chi\left(\left(\Omega^n_{\overline{X}}\right)^{\otimes(1-k)}\right) + E(k) \\
&= \mathrm{Vol}_{\mathrm{HM}}(X) \cdot P_{\check{\mathcal{D}}}(k-1) + E(k)
\end{aligned}$$

and the last equation just uses the definition of the Hilbert polynomial. We describe the error term $E(k)$ in more detail:
A term of

$$Q_n\left((1-k)c_1\left(\Omega^n_{\overline{X}}(\log)\right); c_1\left(\Omega^1_{\overline{X}}\right), \ldots, c_n\left(\Omega^1_{\overline{X}}\right)\right)$$

is of the form (cf. corollary 4.1.18)

$$\left((1-k)c_1\left(\Omega^1_{\overline{X}}(\log)\right)\right)^l \cdot c^\alpha\left(\Omega^1_{\overline{X}}(\log)\right)$$

for some multi-index α with $|\alpha| + l = n$. Since $c_1\left(\Omega^1_{\overline{X}}(\log)\right) = c_1\left(\Omega^n_{\overline{X}}(\log)\right)$ is the pullback of an ample divisor H on the Baily-Borel compactification $\overline{X}^{\mathrm{BB}}$ by lemma 4.2.3, we see:
If $l > \dim(\partial\overline{X}^{\mathrm{BB}})$, the class of H^l can be represented by a cycle with support only on X, as the codimensions cannot add up to yield proper intersection, and the same is true for $c_1\left(\Omega^1_{\overline{X}}(\log)\right)^l$, so we have the equality of intersection products

$$\left(c_1\left(\Omega^1_{\overline{X}}(\log)\right)\right)^l \cdot c^\alpha\left(\Omega^1_{\overline{X}}(\log)\right) = \left(c_1\left(\Omega^1_{\overline{X}}(\log)\right)\right)^l \cdot c^\alpha\left(\Omega^1_{\overline{X}}\right):$$

By lemma 4.1.24, we have

$$c_1\left(\Omega^1_{\overline{X}}\right) + \sum_{D \in \Delta} [D] = c_1\left(\Omega^1_{\overline{X}}(\log)\right)$$

and any intersection of a cycle $[D]$ for $D \subseteq \overline{X} \setminus X$ with a cycle class with a representative supported only on X vanishes. We see that the contributions of these terms in $E(k)$ cancel each other and the remaining polynomial is of degree at most $\dim(\partial\overline{X}^{\mathrm{BB}})$. □

One can describe the individual terms of $E(k)$ as follows:

Proposition 4.2.6. *The error term $E(k)$ is of the form*

$$E(k) = \sum_{i=0}^{s} k^i c_1 \left(\Omega^1_{\overline{X}}(\log)\right)^i \sum_{\substack{|\alpha|+|\beta|=n-i \\ |\beta|\neq 0}} a_{i,\alpha,\beta}\, c^\alpha \left(\Omega^1_{\overline{X}}(\log)\right) \Delta^\beta$$

with universal (i.e. not depending on $\overline{X}$) coefficients $a_{i,\alpha,\beta} \in \mathbb{Q}$ and $s = \dim\left(\partial\overline{X}^{BB}\right)$.

Proof. Consider the more general context of an ample line bundle $\mathcal{L}$ on $\overline{X}$ with

$$\left(c_1\left(\mathcal{L}\right)\right)^l \cdot c^\alpha \left(\Omega^1_{\overline{X}}(\log)\right) = \left(c_1\left(\mathcal{L}\right)\right)^i \cdot c^\alpha \left(\Omega^1_{\overline{X}}\right)$$

for $i > 1$ and define

$$\begin{aligned} E(\mathcal{L}, k) \quad = \quad & Q\left((1-k)c_1\left(\mathcal{L}\right); c_1\left(\Omega^1_{\overline{X}}\right), \ldots, c_n\left(\Omega^1_{\overline{X}}\right)\right) \\ & -Q\left(c_1(\mathcal{L}); c_1\left(\Omega^1_{\overline{X}}(\log)\right), \ldots, c_n\left(\Omega^1_{\overline{X}}(\log)\right)\right). \end{aligned}$$

Using the first equation and proposition 4.2.6, we write this as

$$\begin{aligned} E(\mathcal{L}, k) \quad & = \sum_{i=0}^{n} (1-k)^i (c_i(\mathcal{L}))^i \sum_{|\alpha|+i=n} b_\alpha \left(c^\alpha\left(\Omega^1_{\overline{X}}\right) - c^\alpha\left(\Omega^1_{\overline{X}}(\log)\right)\right) \\ & = \sum_{i=0}^{1} (1-k)^i (c_i(\mathcal{L}))^i \sum_{\substack{|\alpha|+i=n \\ \alpha\neq 0}} b_\alpha \left(c^\alpha\left(\Omega^1_{\overline{X}}\right) - c^\alpha\left(\Omega^1_{\overline{X}}(\log)\right)\right). \end{aligned}$$

Substituting $\mathcal{L} = \Omega^1_{\overline{X}}$ and using the relation lemma 4.1.24 allows us to rewrite the product of Chern classes $c^\alpha\left(\Omega^1_{\overline{X}}(\log)\right)$ as a universal (depending on the dimension n and the number of divisors) sum of products of the (elementary-symmetric polynomials in the) boundary components $D_i \in \Delta$ and the Chern classes of $\Omega^1_{\overline{X}}$, so

$$c^\beta\left(\Omega^1_{\overline{X}}(\log)\right) = \sum_{|\alpha|+|\beta|=n-i} c_{i,\alpha,\beta} c^\alpha\left(\Omega^1_{\overline{X}}\right)\Delta^\beta.$$

Reordering yields the claimed result. □

This may yield a feasible way to compute the error term by splitting it into simpler parts and compute these by various means.
However, this approach ignores the nature of the error term as the natural difference of an Euler characteristic and a logarithmic version of the Euler characteristic. The next section will explore this further.

4.3. Functorial description of the error term

In [Fio17] Fiori gives a systematic and more general framework for dealing with differences of Euler characteristics and their logarithmic counterparts. This section follows closely the treatment there. Note that this is mostly computations with formal polynomials and hence largely independent of the geometric meaning of the objects.
We formalize the notion of *logarithmic Euler characteristics*:

Definition 4.3.1. Let X be a smooth projective variety and Δ a set of simple normal crossing divisors on X. For any sheaf $\mathcal{F}$ on X we define its *logarithmic Euler characteristic* as follows: Let $\Omega^1_X(\log\Delta) = \Omega^1_X(\log)$ be the sheaf of Δ-logarithmic 1-forms on X. The *logarithmic Euler characteristic* of $\mathcal{F}$ with respect to Δ is

$$\chi(X,\Delta,\mathcal{F}) = Q_n\left(c_0(\mathcal{F}), c_1(\mathcal{F}), \ldots, c_n(\mathcal{F}); c_1\left(\Omega^1_X(\log)\right), \ldots, c_n\left(\Omega^1_X(\log)\right)\right)$$

with Q_n the universal polynomial from corollary 4.1.18.

Since we will be working with a fixed variety X and fixed collection Δ of divisors, we will often refer to the Chern classes of Ω^1_X simply as *the* Chern classes and of the Chern classes of $\Omega^1_X(\log)$ as *the* logarithmic Chern classes.
The quantity of interest to us is of course the difference

$$\chi(X,\Delta,\mathcal{F}) - \chi(X,\mathcal{F})$$

as this is the natural generalization of the error term encountered in theorem 4.2.5.
To give better descriptions of this difference, we introduce some further notations. We will mostly follow [Fio17]: Let

$$\Delta = \{D_i |\, i = 1,\ldots l\}$$

and any multi-index $\underline{b} = (b_1,\ldots,b_l) \in \mathbb{Z}^l_{\geq 0}$: We define

$$[D^{\underline{b}}] = \prod_{i=1}^{l} [D_i]^{b_i}$$

to be the formal product of the divisors or the intersection product in the Chow ring of X whenever this makes sense. The *degree* of $\underline{b}$ is

$$|\underline{b}| = \sum_{i=1}^{l} b_i.$$

Using the multiplicativity (cf. [Fio17]) of the universal polynomials one gets an alternative representation of the difference $\chi(X,\Delta,\mathcal{F}) - \chi(X,\mathcal{F})$ as a weighted sum of easier objects.

Proposition 4.3.2 ([Fio17, Theorem 4.1]). *There are constants $\delta_{\underline{b}} \in \mathbb{Q}$, such that for any $X, \Delta, \mathcal{F}$ as above, the difference*

$$\chi(X, \Delta, \mathcal{F}) - \chi(X, \mathcal{F})$$

equals

$$\sum_{|\underline{b}|\geq 1} (-1)^{|\underline{b}|}\delta_{\underline{b}}[D^{\underline{b}}]Q_{n-|\underline{b}|}\left(c_0(\mathcal{F}), \dots, c_{n-|\underline{b}|}(\mathcal{F}); c_1\left(\Omega^1_X(\log)\right), \dots, c_{n-|\underline{b}|}\left(\Omega^1_X(\log)\right)\right).$$

Remark 4.3.3. The constants $\delta_{\underline{b}}$ are universal in the sense that they depend only on the dimension n.

If every entry b_i of $\underline{b}$ is at most 1, $D^{\underline{b}}$ is *multiplicity free*. In this case, the intersection product does not involve any self-intersection.
Fortunately, logarithmic Chern classes behave very well with regard to the restriction to elements of Δ:

Proposition 4.3.4 ([Fio17, Proposition 2.11]). *Let X, Δ be as before. Fix an irreducible $D \in \Delta$ with inclusion $i : D \hookrightarrow X$, then*

$$[D] \cdot c^{\alpha}\left(\Omega^1_x(\log\Delta)\right) = i_* c^{\alpha}\left(\Omega^1_D\left(\log\Delta'\right)\right)$$

with $\Delta' := \Delta \cap D = \{D_i \cap D |\, D_i \in \Delta$ with $D \nsubseteq D_i\}$ and for any multi-index α.

We give a self-contained proof, reproducing the one from [CMZ20, Section 2]:

Proof. The ideal sheaf sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0$$

and the multiplicativity of the total Chern class on short exact sequences yield

$$1 = c(\mathcal{O}_X(-D))c(i_*\mathcal{O}_D) = (1 - [D])\, c(i_*\mathcal{O}_D),$$

so

$$c_1(i_*\mathcal{O}_D) = \frac{1}{1-[D]};$$

moreover, the normal bundle $\mathcal{N}_D$ to D is given by the pullback $i^*\mathcal{O}_X(D)$ and we get

$$c(\mathcal{N}_D) = c(i^*\mathcal{O}_X(D)) = 1 + i^*[D]$$

The conormal bundle sequence

$$0 \to \mathcal{N}^*_D \to i^*\Omega^1_X \to \Omega^1_D \to 0$$

then implies

$$c(\Omega_D) = i^*\left(c\left(\Omega^1_X\right)\right)c(\mathcal{N}^*_D)^{-1} = i^*\left(c\left(\Omega^1_X\right)\left(1 - [D]^{-1}\right)\right).$$

We now consider the logarithmic cotangent bundle with respect to Δ : The fundamental sequence

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log \Delta) \longrightarrow \bigoplus_j (i_j)_* \mathcal{O}_{D_j} \longrightarrow 0$$

shows

$$c\left(\Omega^1_X(\log \Delta)\right) = c\left(\Omega^1_X\right) \prod_j \frac{1}{1-[D_j]}$$

and a further application to $\Omega^1_D(\log \Delta')$ gives

$$c\left(\Omega^1_D(\log \Delta')\right) = c\left(\Omega^1_D\right) \prod_{D_j \neq D} \frac{1}{1-[D_j \cap D]}.$$

This enables us to compute

$$\begin{aligned} c(\Omega^1_D(\log \Delta')) &= i^*\left(c\left(\Omega^1_X\right)(1-[D])^{-1}\right) \cdot \prod_{D_j \neq D} \frac{1}{1-[D_j \cap D]} \\ &= i^*\left(c\left(\Omega^1_X\right)(1-[D])^{-1}\right) \cdot i^*\left(\prod_{D_j \neq D} \frac{1}{1-[D_j]}\right) \\ &= i^*\left(c\left(\Omega^1_X\right) \prod_j \frac{1}{1-[D_j]}\right) \\ &= i^*\left(c\left(\Omega^1_X(\log \Delta)\right)\right). \end{aligned}$$

Hence the claimed equality is true for the Chern classes and therefore also for their products. □

For multiplicity-free $\underline{b}$ this allows us to give geometric meaning to products of the form

$$[D^{\underline{b}}]Q_{n-|\underline{b}|}\left(c_0(\mathcal{F}), \ldots, c_{n-|\underline{b}|}\left(\Omega^1_X(\log \Delta)\right)\right):$$

Lemma 4.3.5 ([Fio17, Notation 4.2])**.** *Let $\underline{b}$ be multiplicity-free. Let C_j be the irreducible components of the intersection product $[D^{\underline{b}}]$, so*

$$[D^{\underline{b}}] = \sum_i [C_i].$$

Then

$$\begin{aligned} & \chi\left(D^{\underline{b}}, \Delta \cap D^{\underline{b}}, \mathcal{F}|_{D^{\underline{b}}}\right) \\ := & \sum_j \chi\left(C_j, \Delta \cap D^{\underline{b}}, \mathcal{F}|_{D^{\underline{b}}}\right) \\ = & [D^{\underline{b}}]Q_{n-|\underline{b}|}\left(c_0(\mathcal{F}), \ldots, c_{n-|\underline{b}|}(\mathcal{F}); c_1\left(\Omega^1_X(\log)\right), \ldots, c_{n-|\underline{b}|}\left(\Omega^1_X(\log)\right)\right) \end{aligned}$$

This can be used to interpret some parts of proposition 4.3.2 as actual Euler characteristics:

Proposition 4.3.6 ([Fio17, Corollary 4.5])**.** *Let $X, \Delta, \mathcal{F}$ as in proposition 4.3.2. There are constants $\lambda_{\underline{b}}$ such that*

$$\begin{aligned}\chi(X,\Delta,\mathcal{F})-\chi(X,\mathcal{F}) &= \sum_{\substack{|\underline{b}|\geq 1\\ \textit{multiplicity free}}} \lambda_{\underline{b}}\chi\left(D^{\underline{b}},\mathcal{F}|_{D^{\underline{b}}}\right)\\ &+ \sum_{\substack{|\underline{b}|\geq 1\\ \textit{non-multiplicity free}}} \lambda_{\underline{b}}\chi\left(D^{\underline{b}},\Delta,\mathcal{F}|_{D^{\underline{b}}}\right)\end{aligned}$$

where $\chi\left(D^{\underline{b}},\Delta,\mathcal{F}|_{D^{\underline{b}}}\right)$ is just a shorthand for

$$[D^{\underline{b}}]Q_{n-|\underline{b}|}\left(c_0(\mathcal{F}),\ldots,c_{n-|\underline{b}|}(\mathcal{F});c_1\left(\Omega^1_X(\log)\right),\ldots,c_{n-|\underline{b}|}\left(\Omega^1_X(\log)\right)\right).$$

Remark 4.3.7*.* Again, the constants $\lambda_{\underline{b}}$ depend only on the dimension n. Moreover, $\lambda_{\underline{b}}$ depends only on the multiset

$$\{b_i\in\underline{b}|\, b_i\neq 0\}\,.$$

Fiori provides some helpful tools to compute these numbers:

i) If $b_i\in\{0,1\}$ for all entries in $\underline{b}$, then

$$\lambda_{\underline{b}}=(-1)^{|\underline{b}|}\delta_{\underline{b}},$$

cf. [Fio17, Proposition 2.8]

ii) If $b_i=1$ for at least one entry of $\underline{b}$ and $\underline{b}$ has another entry b_j with $b_j\geq 2$, then

$$\lambda_{\underline{b}}=0$$

cf. [Fio17, Proposition 2.9]

iii) If $b_i\geq 2$ for all non-vanishing b_i, then

$$\lambda_{\underline{b}}=\delta_{\underline{b}}$$

Table 4.1 gives the values of the non-vanishing $\lambda_{\underline{b}}$ for all possibilities up to $n=10$; the multiset $(a,a,\ldots,a)$ is denoted by a^b.

To handle $[D^{\underline{b}}]Q_{n-|\underline{b}|}(\ldots)$ for non-multiplicity-free $\underline{b}$, Fiori provides tools as well. Note that, while $[D^{\underline{b}}]$ denotes a product of possibly many different elements of Δ, $[D^b]$ just denotes the b-fold intersection product of a single class $[D]$ for $D\in\Delta$ with itself. For ease of notation we will denote

$$Q_m\left(c_0(\mathcal{F}),\ldots,c_m(\mathcal{F});c_0\left(\Omega^1_X(\log\Delta)\right),\ldots,c_k\left(\Omega^1_X(\log\Delta)\right)\right)$$

Degree	$\underline{b}$	$\lambda_{\underline{b}}$
1	(1)	$\frac{1}{2}$
2	(1^2)	$\frac{-1}{4}$
	(2)	$\frac{1}{12}$
3	(1^3)	$\frac{1}{8}$
4	(1^4)	$\frac{-1}{16}$
	$(2,2)$	$\frac{1}{144}$
	(4)	$\frac{-1}{720}$
5	(1^5)	$\frac{1}{32}$
6	(1^6)	$\frac{-1}{64}$
	(2^3)	$\frac{1}{1728}$
	$(2,4)$	$\frac{-1}{8640}$
	(6)	$\frac{1}{30240}$
7	(1^7)	$\frac{1}{128}$
8	(1^8)	$\frac{-1}{256}$
	(2^4)	$\frac{1}{20736}$
	$(2^2,4)$	$\frac{-1}{103680}$
	(4^2)	$\frac{1}{518400}$
	$(2,6)$	$\frac{1}{362880}$
	(8)	$\frac{-1}{1209600}$
9	(1^9)	$\frac{1}{512}$
10	(1^{10})	$\frac{-1}{1024}$
	(2^5)	$\frac{1}{248832}$
	$(2^3,4)$	$\frac{-1}{1244160}$
	$(2,4^2)$	$\frac{1}{6220800}$
	$(2^2,6)$	$\frac{1}{4354560}$
	$(4,6)$	$\frac{-1}{21772800}$
	$(2,8)$	$\frac{-1}{14515200}$
	(10)	$\frac{1}{47900160}$

Table 4.1.: $\lambda_{\underline{b}}$ for small $|\underline{b}|$

simply by

$$Q_m\left(\mathrm{c}(\mathcal{F});\mathrm{c}\left(\Omega^1_X(\log\Delta)\right)\right).$$

We describe the behavior of logarithmic Chern classes under restriction to divisors not in Δ:

Lemma 4.3.8. *Let X,Δ be as before and let E be a closed subscheme of X with morphism $i_E : E \hookrightarrow X$ such that E intersects all elements of Δ transversely. Then: There are constants $\delta_{(k-1)} \in \mathbb{Q}$ (depending only on m) such that*

$$[E]\cdot\alpha\cdot Q_m\left(\mathrm{c}(\mathcal{F});\mathrm{c}\left(\Omega^1_X(\log\Delta)\right)\right)$$

equals

$$\begin{aligned}&(i_E)_*\left[i_E^*(\alpha)\cdot Q_m\left(\mathrm{c}(i_E^*(\mathcal{F}));\mathrm{c}\left(\Omega^1_E(\log\Delta')\right)\right)\right]\\ -&\sum_{k=1}^{m}\delta_{(k)}\alpha[E]^{k+1}Q_{m-k}\left(\mathrm{c}(\mathcal{F});\mathrm{c}\left(\Omega^1_X(\log\Delta)\right)\right)\end{aligned}$$

where $\Omega^1_D(\log\Delta')$ denotes the logarithmic cotangent bundle with respect to the collection

$$\Delta' = \{D\cap E|\, D\in\Delta\}$$

and $\alpha\in\mathrm{CH}(\overline{X}^{tor}_\Sigma)$ is an arbitrary cycle class.

Note that this differs in notation from the statement in [Fio17]. This is mainly due to the fact that the notation $\Omega^1_X(\log\Delta\cdot E)$ there seems to be not well-defined: The elements of $\Delta\cdot E$ are of the form $D\cap E$ for $D\in\Delta$ and therefore of codimension 2 in X. As this collection does not constitute a normal crossing divisor, the meaning of $\Omega^1_X(\log\Delta\cdot E)$ is unclear.
To avoid this discrepancy, we sketch a proof for our version of this lemma:

Proof. A computation analogous to the one in the proof of proposition 4.3.4 shows the equality

$$\begin{aligned}c\left(\Omega^1_E(\log\Delta')\right) &= i_E^*\left(c\left(\Omega^1_X(\log\Delta)\right)\cdot(1+[E])^{-1}\right)\\ &= i_E^*\left(c\left(\Omega^1_X(\log\Delta)\right)\cdot\left(1+[E]+[E]^2+\ldots\right)\right).\end{aligned}$$

This translates to the Chern classes as

$$c_i\left(\Omega^1_E(\log\Delta')\right)=\sum_{j+k=i} i_E^*\left(c\left(\Omega^1_X(\log\Delta)\right)\right)\cdot i_E^*[E]^k$$

and an application of [Fio17, Theorem 4.1] together with repeated use of the projection formula of lemma 4.1.4 yields the result. $\square$

The constant $\delta_{(k-1)}$ is just $\delta_{\underline{b}}$ from proposition 4.3.2 for the single-element multiset $\underline{b} = \{k-1\}$ and agrees with $-\lambda_{\{k-1\}}$. For $k-1 \leq 10$ these values can be found in table 4.1.
We come back to the description of

$$E(k) = \chi\left(\overline{X}, \Delta, \Omega^n_{\overline{X}}(\log)^{\otimes(1-k)}\right) - \chi\left(\overline{X}, \Omega^n_{\overline{X}}(\log)^{\otimes(1-k)}\right)$$

that sparked this considerations. Proposition 4.3.6 showed that $E(k)$ can be written as the sum of actual Euler characteristic on the irreducible components of multiplicity-free intersection products and non-multiplicity-free terms of the form

$$[D]^{\underline{b}} \cdot Q_m\left(\mathrm{c}(\mathcal{F}); \mathrm{c}\left(\Omega^1_X(\log\Delta)\right)\right).$$

The latter can be handled by deforming the self-intersection products to proper intersections via suitable relations in the Chow ring $\mathrm{CH}(X)$. The preceding relations will be helpful with this:
An intersection number

$$[D]^{\underline{b}} \cdot [E] \cdot Q_m\left(\mathrm{c}(\mathcal{F}); \mathrm{c}\left(\Omega^1_X(\log\Delta)\right)\right)$$

with multiplicity-free $\underline{b}$ and $D_i \in \Delta$ can, by the preceding lemma, be computed as

$$\chi(Y, \Delta', \mathcal{F}|_Y) + \text{rest term},$$

where $Y = \Delta^{\underline{b}} \cap E$, $\Delta' = \Delta \cap \Delta^{\underline{b}} \cap E$ and the rest term consisting of products of self-intersections of $E \cap D^{\underline{b}}$ and products of logarithmic Chern classes. We note:

i) The Euler characteristic is a geometric invariant depending only on Y, which is of smaller dimension than X and should be easier to understand.

ii) The indices of the universal polynomials in the rest term are strictly smaller than the one we began with, so we decrease the number of logarithmic Chern classes appearing in the computation by gaining additional products of classes of divisors. Since the latter carry more direct geometric meaning than the first, they will be easier to handle.

Moreover, by noting the decrease in dimension of the underlying geometric object resp. the index of the universal polynomial, a repeated use of this consideration allows an inductive treatment. We will make use of this in chapter 13.
This concludes this first and most general part of this thesis. The next part will be concerned with the more specialized case of orthogonal locally symmetric spaces corresponding to lattices.

Part II.

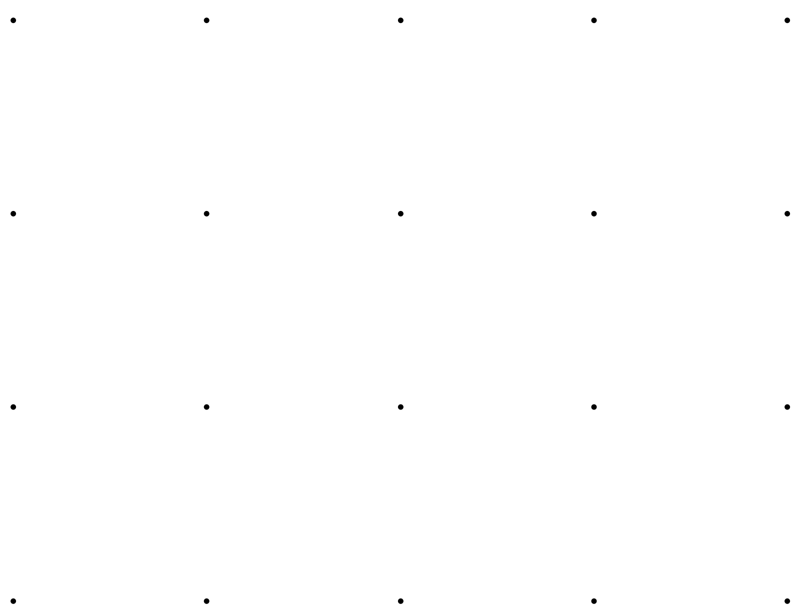
Orthogonal locally symmetric spaces

5. Lattices and orthogonal groups

This second part of this thesis is concerned with the application of the general theory of the last part to the case of (locally) symmetric spaces of orthogonal type.
In this chapter we give a short introduction into lattices and their automorphism groups. These will be exactly the semisimple linear algebraic groups inducing the locally symmetric spaces of interest.
Integral lattices together with their structure theory and their automorphism groups are of vital importance in a number of different area of mathematics. We will introduce these concepts and the fundamental results about them in the course of this chapter. Good general references for these topics are the fundamental article [Nik79] and the classic book [CS98].

5.1. Lattices and discriminant forms

General lattices are the formalization and generalization of additive subgroups of $\mathbb{R}^n$ which are isomorphic to $\mathbb{Z}^n$. For example, one can think of the set of points in $\mathbb{R}^2$ with integral coordinates. The picture is well-known and as follows:



These objects generalize to the following notion:

Definition 5.1.1. An integral *lattice* L (or short: *lattice*) is a free $\mathbb{Z}$-module of finite rank n equipped with a non-degenerate bilinear form $(\cdot,\cdot) : L \times L \to \mathbb{Z}$. The bilinear form $(\cdot,\cdot)$ induces a quadratic form by

$$q(\lambda) := \frac{1}{2}(\lambda,\lambda) \in \mathbb{Z} \qquad \text{for all } \lambda \in L.$$

A *sublattice* is a submodule $M \subseteq L$ with the restricted bilinear form. The integral lattice L is called *even* if $(\lambda,\lambda) \in 2\mathbb{Z}$ for all $\lambda \in L$ or equivalently if q takes its values in $\mathbb{Z}$. By classic abuse of notation we will often neglect the bilinear map, and just speak of the lattice L.
Assume that L is non-degenerate with respect to $(\cdot,\cdot)$. Define $V := L \otimes_{\mathbb{Z}} \mathbb{R}$. The bilinear form $(\cdot,\cdot)$ on $L \times L$ extends to an $\mathbb{R}$-bilinear form on $V \times V$, as does q. We will abuse notation and denote these extensions by the same notation $(\cdot,\cdot)$ resp. q regardless of the space they are defined on. The pair (V,q) is called a *quadratic space*.

Definition 5.1.2. An *isomorphism* between lattices L_1 and L_2 with bilinear form $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ is a $\mathbb{Z}$-linear invertible map $\varphi : L_1 \to L_2$ with

$$(x,y)_1 = (\varphi(x),\varphi(y))_2$$

for all $x,y \in L_1$. The lattices L_1, L_2 are called *isomorphic*, denoted $L_1 \cong L_2$ if there exists an isomorphism $L_1 \to L_2$.

By the famous *inertia law* of Sylvester, any non-degenerate symmetric bilinear form B on a n-dimensional $\mathbb{R}$-vector space can be diagonalized over $\mathbb{Q}$. Denote by k the number of positive eigenvalues and by $n-k$ the number of negative eigenvalues. The pair $(k, n-k)$ is called the *signature* of B. For a lattice L with quadratic form q, the signature $\operatorname{sig}(L)$ of L is the signature of the extended bilinear form $(\cdot,\cdot)$ on $V = L \otimes \mathbb{R}$. If k or $n-k$ is 1, we call the lattice *Lorentzian*. We denote by $\operatorname{sgn}(L)$ the value $n-2k$.
The value $q(\lambda)$ of the quadratic form on $\lambda \in L$ is called the *norm* of λ, a vector λ with $q(\lambda) = 0$ is called *isotropic*. Moreover, we note that a quadratic form $q : L \to \mathbb{Z}$ defines a bilinear form b on $L \times L$ by the *polarization identity*

$$b(u,v) = (q(u+v) - q(u) - q(v)),$$

so we will often define a lattice only by given a quadratic form. This works more generally over any field of characteristic $\neq 2$. By choosing a basis $B = (e_1, \ldots, e_n)$ of L over $\mathbb{Z}$, we can associate to any lattice its *Gram matrix* $A_{L,B}$ by $(A_{L,B})_{i,j} = (e_i, e_j)$. The condition for integrality of L is equivalent to $A_{L,B}$ having integral entries, while the evenness of L translates to the diagonal entries being in $2\mathbb{Z}$. Obviously, these properties do not depend on the chosen basis of L.

Definition 5.1.3. Given two lattices L_1, L_2 with bilinear forms $(\cdot,\cdot)_1, (\cdot,\cdot)_2$ the *orthogonal direct sum* is the lattice with $\mathbb{Z}$-module $L_1 \oplus L_2$ and bilinear form defined by

$$((x_1, x_2), (y_1, y_2)) = (x_1, y_1) + (x_2, y_2).$$

We say that a lattice L splits as $L_1 \oplus L_2$ if $L \cong L_1 \oplus L_2$.

Definition 5.1.4. The *orthogonal group* of the quadratic space (V, q) is the subgroup $\mathrm{O}(q)$ of automorphisms of V preserving q. We denote the connected component of the identity by $\mathrm{O}^+(q)$ and the subgroup of elements with determinant $+1$ by $\mathrm{SO}(q)$. The *orthogonal group* $\mathrm{O(L)}$ of L is the subgroup of elements of $\mathrm{O}(q)$ preserving L; its intersection with the aforementioned subgroups are denoted $\mathrm{O}^+(L)$ resp. $\mathrm{SO}(L)$. Analogously, we define

$$\mathrm{SO}^+(q) = \mathrm{SO}(q) \cap \mathrm{O}^+(q) \text{ resp. } \mathrm{SO}^+(L) = \mathrm{SO}(L) \cap \mathrm{O}^+(q).$$

Note that, by Sylvester's law of inertia, the orthogonal group $\mathrm{O}(q)$ depends only on the signature of q, so we will also use the notation $\mathrm{O}(k, n-k)$ for $\mathrm{O}(q)$.

Remark 5.1.5. In view of the discussion about linear algebraic groups in chapter 1 we note that the orthogonal groups of quadratic spaces that arise from integral lattice are obviously linear algebraic groups and the non-degenerateness implies that these groups are semisimple.

Definition 5.1.6. The *dual lattice* L' of a lattice L is

$$L' = \{\mu \in V \,|\, (\mu, \lambda) \in \mathbb{Z} \text{ for all } \lambda \in L\}.$$

By construction we have $L \subseteq L'$. If $L = L'$, the lattice is called *unimodular*. The *level* of L is

$$\mathrm{Lev}(L) = \min\{n \in \mathbb{N} \,|\, nq(\lambda) \in \mathbb{Z} \text{ for all } \lambda \in L'\}.$$

One can easily show that a lattice is unimodular if and only if every Gram matrix (or, equivalently, any of its Gram matrices) has determinant ± 1.

Example 5.1.7. For our purposes, the two most important lattices are the *hyperbolic plane* and the E_8-lattice.

- The hyperbolic plane is given by the free $\mathbb{Z}$-module $\mathbb{Z}^2$ equipped with the quadratic form given by $q(x, y) = xy$ or $q(x, y) = x^2 - y^2$. These quadratic forms determine the same lattice structure on $\mathbb{Z}^2$ as they can be transformed into each other by the change of basis $(u, v) \mapsto (u + v, u - v)$. Choosing the standard basis of $\mathbb{Z}^2$ we see that the bilinear form b is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

  so the hyperbolic plane is an integral and even rank 2-lattice with signature $(1, 1)$.

- The second example is the E_8-lattice. Its simplest model is as follows: Consider the $\mathbb{Z}$-module

$$\mathrm{E}_8 = \left\{ x = (x_1, \ldots, x_8) \in \mathbb{Z}^8 \cup \left(\frac{1}{2} + \mathbb{Z}\right)^8 : \sum_{i=1}^{8} x_i \in 2\mathbb{Z} \right\} \subseteq \mathbb{R}^8$$

  with the bilinear form induced by the the usual inner product on $\mathbb{R}^8$. A basis of E_8 is given by the columns of

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix}$$

  and the corresponding Gram matrix is given by

$$\begin{pmatrix} 4 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

  and we see that the lattice is integral and even, moreover one can compute that it is unimodular.

It is possible to classify integral lattices: The methods are described in [CS98] and in part in the subsequent section. This classification shows that these two lattices are quite extraordinary. Up to a isomorphism, these are the only even and integral lattices of their respective ranks. Moreover, even unimodular positive definite lattices are quite rare as they only exist in dimensions that are multiples of 8. For even unimodular lattices (without any definiteness condition) one can show:

Proposition 5.1.8. *Let $(p, q) \in \mathbb{Z}^2$. There exists an even unimodular lattice of rank $p + q$ and signature (p, q) if and only if $8 | p - q$. If $pq \neq 0$, this lattice is unique up to isomorphism.*

The last condition is strictly necessary for uniqueness: If $q = 0$ and $p = 8k \geq$ the number of even unimodular lattices grows rapidly. For $p = 16$ there are two non-isomorphic even unimodular lattice (the orthogonal sum E_8^2 and a lattice denoted by D_{16}^+), for $p = 24$

there are 24 of them (23 so-called *Niemeier* lattices and the Leech lattice Λ singled out by having no vectors of norm 2) and for $p = 32$ the number is known to be larger than one billion.
To classify lattices, the following concept of *discriminant forms* is useful.

Definition 5.1.9. Let D be a finite abelian group and $q : D \to \mathbb{Q}/\mathbb{Z}$ a map (called *finite quadratic form*) with

i) $q(a\gamma) = a^2 q(\gamma)$ for $a \in \mathbb{Z}$ and $\gamma \in D$

ii) the map $D \times D \to \mathbb{Q}/\mathbb{Z}$ given by $(\gamma, \delta) = q(\gamma + \delta) - q(\gamma) - q(\delta)$ is $\mathbb{Z}$-bilinear and non-degenerate.

The pair (D, q) is called a *discriminant form.*
Two elements $\gamma_1, \gamma_2 \in D$ are *orthogonal* if $(\gamma_1, \gamma_2) = 0 + \mathbb{Z}$. If $\gamma \in D$ is orthogonal to itself, so $q(\gamma) = 0 + \mathbb{Z}$, it is called *isotropic.* A subgroup $H \subseteq D$ is called isotropic if every $\gamma \in H$ is isotropic. An isomorphism of discriminant forms is a group isomorphism respecting the finite quadratic forms. The group of automorphisms of D is denoted by $\mathrm{O}(D)$.

As for lattices one can construct orthogonal direct sums: For discriminant forms D_1, D_2 with finite quadratic forms q_1, q_2 set $D = D_1 \oplus D_2$ and $q(x_1, x_2) = q_1(x_1) + q_2(x_2)$ for $x_1 \in D_1, x_2 \in D_2$. A discriminant form D *splits* as $D_1 \oplus D_2$ if D is isomorphic to $D_1 \oplus D_2$ as discriminant forms.
There is a natural way of constructing discriminant forms from even lattices:
The *discriminant group* of L is the finite abelian group $\Delta_L = L'/L$. If L is even, the bilinear and quadratic form descend to $\mathbb{Q}/\mathbb{Z}$-valued forms on the discriminant group and the corresponding structure is a discriminant form in the sense of definition 5.1.9.
Moreover, the converse is true as well: For every discriminant form D there is an even lattice L with $D \cong L'/L$.
A discriminant form D is *decomposable* if there are non-trivial discriminant forms D_1, D_2 with $D \cong D_1 \oplus D_2$. There is a complete structure theory of discriminant forms. The most important indecomposable discriminant forms for our purposes are the following:

Definition 5.1.10. Let $q > 1$ be the power of an odd prime p, $D = \mathbb{Z}/q\mathbb{Z}$ and $\epsilon = \pm 1$. For any generator $\gamma \in D$ define $q : D \to \mathbb{Q}/\mathbb{Z}$ by $q(\gamma) = a/q + \mathbb{Z}$ with $\left(\frac{2a}{p}\right) = \epsilon$ (Legendre symbol). The corresponding discriminant form is called the *indecomposable p-adic Jordan component of order* q, denoted by q^ϵ. Let $q^{\epsilon_1}, \ldots, q^{\epsilon_n}$ be n indecomposable p-adic Jordan components of order q and let $\epsilon = \epsilon_1 \cdot \ldots \cdot \epsilon_n$. The direct sum

$$q^\epsilon = \bigoplus_{i=1}^{n} q^{\epsilon_i}$$

is called the *p-adic Jordan component of rank n and order q.*

These discriminant forms are the building blocks of general discriminant forms in the following sense.

Proposition 5.1.11. *Let L be an even non-degenerate lattice of odd prime level p. The discriminant form of L decomposes uniquely into a direct sum of p-adic Jordan components.*

The decomposition of Jordan components into indecomposable Jordan components is no longer unique as, for example, $q^{+2} \cong q^{+} \oplus q^{+} \cong q^{-} \oplus q^{-}$.
For lattices with even level there are more possible types of (indecomposable) Jordan components, cf. [Sch06, Section 3]. Since we are only dealing with lattices of odd level, we will not encounter them.
Discriminant forms are a useful tool for describing the *even overlattices* for a given even lattice (an even M being an even overlattice of an even L if and only if L is a sublattice of M of the same rank):

Proposition 5.1.12 ([Nik79, Proposition 1.4.1])**.** *Let L be an even lattice with discriminant form Δ_L and let M be an even overlattice of L with discriminant form Δ_M. Then $H_M = M/L \subseteq \Delta_L$ is an isotropic subgroup of Δ_L and the mapping*

$$M \mapsto H_M$$

is a bijection between even overlattices of L and isotropic subgroups of Δ_L. This correspondence identifies Δ_M with $(H_M)^{\perp}_{\Delta_L}/H_M \subseteq D_L$.

In particular:

Corollary 5.1.13. *Let L be an even unimodular lattice and let M be an even lattice of the same signature and with discriminant form $\Delta_M = q^{\epsilon n}$ for $q > 1$ power of an odd prime. Then M can be considered as a sublattice of L if and only if Δ_M contains an isotropic subgroup of rank $n/2$.*

Proof. This follows easily from proposition 5.1.12: Since Δ_L is trivial, we have to find an isotropic subgroup $H \subseteq \Delta_M$ with $(H)^{\perp}_{\Delta_M}/H$ trivial, but this is equivalent to H being of rank $d/2$ as

$$|(H)^{\perp}_{\Delta_M}/H| = |\Delta_M|/|H|^2.$$

□

There is one more invariant of discriminant forms:

Definition 5.1.14. Let D be a discriminant form. The signature of D is the signature of any even non-degenerate lattice L with $D \cong L'/L$ as discriminant forms, computed modulo 8.

This is well-defined and independent of the chosen lattice by *Milgram's formula*, cf. [MH73, Appendix 4].
These two concepts of discriminant forms and signatures allow the already mentioned classification of even lattices. The *genus* of an even non-degenerate lattice L of signature (b_+, b_-) is the equivalence class of isomorphism classes of all lattices with discriminant form $D = L'/L$ and signature (b_+, b_-), denoted by $I\!I_{b_+,b_-}(D)$.

In general, there are several non-isomorphic lattices with the same genus, albeit their number is finite.
A sufficient condition for uniqueness in the genus is given by the following lemma:

Lemma 5.1.15 ([CS98, Corollary 15.22])**.** *Let L be an indefinite lattice of rank n. If*

$$|\Delta_L| < 5^{\binom{n}{2}},$$

then there is only one isomorphism class of lattices in its genus.

Whenever we use the genus symbol to denote the lattice, it is implicit that this lattice is unique in its genus.

5.2. Orthogonal groups

We turn our attention to the automorphism groups of lattices and discriminant forms. Note that $\mathrm{O}(L) = \mathrm{O}(L')$, so we get a natural homomorphism

$$\mathrm{O}(L) \to \mathrm{O}(\Delta_L).$$

A natural subgroup of $\mathrm{O}(L)$ is the *discriminant kernel*.

Definition 5.2.1. We denote by $\widetilde{\mathrm{O}}(L)$ the kernel of the map

$$\mathrm{O}(L) \to \mathrm{O}(\Delta_L)$$

and define further

$$\widetilde{\mathrm{SO}}^+(L) = \widetilde{\mathrm{O}}(L) \cap \mathrm{SO}^+(L),$$

the connected component of the identity. This is the *discriminant kernel* of L.

For any sublattice $K \subseteq L$ the discriminant kernel satisfies $\widetilde{\mathrm{SO}}^+(K) \subseteq \widetilde{\mathrm{SO}}^+(L)$.
If the lattices L splits two hyperbolic planes, the discriminant kernel is fairly large:

Lemma 5.2.2 ([Eic52, Section 10] or, in modern language [GHS09, Proposition 3.3])**.** *Let $L = L_0 \oplus II_{1,1} \oplus II_{1,1}$. For primitive $u, v \in L'$ with $q(v) = q(w)$ and $u + L = v + L$ there exists $\phi \in \widetilde{\mathrm{O}}(L)$ with $\phi(u) = v$.*

We give a family of examples:

Example 5.2.3. Given any lattice L and a $N \in \mathbb{N}$, we can define the *scaled lattice* $L(N)$ as the free $\mathbb{Z}$-module L together with the scaled bilinear form

$$(\cdot,\cdot)_{L(N)} = N \cdot (\cdot,\cdot)_L.$$

Considering L as lying inside $V = L \otimes \mathbb{R}$, the scaled lattice is given by $\sqrt{N}L$. The discriminant kernel of $L(N)$ is as follows: Fix a basis of L and consider $\mathrm{O}(L)$ as a subgroup of $\mathrm{GL}_n(\mathbb{Z})$ (or, equivalently, consider any faithful representation $\mathrm{O}(L) \to \mathrm{GL}_n(\mathbb{Z})$). The dual lattice $(L(N))'$ of $L(N) = \sqrt{N}L$ inside V is $1/\sqrt{N}L$ and a short computation shows that $\widetilde{\mathrm{SO}}^+(L(N))$ is the subgroup

$$\{A \in \mathrm{SO}^+(L) \subseteq \mathrm{GL}_n(\mathbb{Z}) |\, A \equiv \mathrm{Id} \bmod N\}.$$

A useful observation on the rescaling of a lattice L by some $N \in \mathbb{Z}$ is the following: While the lattice $L\left(N^2\right)$ has a natural realization as a sublattice of L consisting of the N-multiples of elements of L, the rescaling by an arbitrary (possibly non-square) N may not be considered as a sublattice of L, even if they obviously lie in the same $\mathbb{Q}$-vector space $L(N) \otimes \mathbb{Q} = L \otimes \mathbb{Q}$.
To check whether $L(N) \subseteq L$ holds, one can use corollary 5.1.13. We will do this exemplarily for the case of a rescaling of the direct sum of k E_8-lattices and $2l$ hyperbolic planes:

Example 5.2.4. Let $L = k\mathrm{E}_8 + mII_{1,1}$ for $k \geq 0, m = 2l \geq 2$. Let p be an odd prime. The discriminant group of $L(p)$ is

$$L(p)'/L(p) = \frac{1}{\sqrt{p}}L/\sqrt{p}(L) \cong L/pL \cong (\mathbb{Z}/p\mathbb{Z})^{8k+2m},$$

so the discriminant form is $p^{\epsilon n}$ for $n = 8k + 2m$ while that of L is trivial. By the theory of discriminant forms (cf. *oddity formula*, *p-excess*, see [CS98, Chapter 15]), one can determine ϵ to be $+1$. It remains to construct an isotropic subgroup of rank $4k + m$ in Δ_L, then we get $L(p) \subseteq L$ from corollary 5.1.13: We distinguish the possible residues of p modulo 4:

i) $p \equiv 1 \mod 4$: Then p^+ can be generated by a $\gamma_1 \in p^+$ with $q(\gamma_1) = a/p + \mathbb{Z}$ with

$$+1 = \left(\frac{2a}{p}\right).$$

Since, in this case,

$$\left(\frac{2a}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2a}{p}\right) = \left(\frac{2(-a)}{p}\right),$$

we can find another generator γ_2 with $q(\gamma_2) = (-a)/p + \mathbb{Z}$, hence $p^+ \oplus p^+$ has a subgroup of rank 1 generated by (γ_1, γ_2) with the property that $q(\gamma_1) = a/p + \mathbb{Z}$ and $q(\gamma_2) = (-a)/p + \mathbb{Z}$, so $q(\gamma_1, \gamma_2) = q(\gamma_1) + q(\gamma_2) = 0 + \mathbb{Z}$. Decomposing

$$p^{+n} = \bigoplus_{i=1}^{n/2}(p^+ \oplus p^+)$$

shows that applying this construction to every of the $n/2$ factors and taking the direct sum of the resulting isotropic subgroups yields an isotropic subgroup of the desired rank.

ii) $p \equiv 3 \mod 8$: We note

$$\left(\frac{2a}{p}\right) = -\left(\frac{-1}{p}\right)\left(\frac{2a}{p}\right) = -\left(\frac{2(-a)}{p}\right)$$

in this case, so if p^+ is generated by $\gamma_1 \in p^+$ with $q(\gamma_1) = a/p + \mathbb{Z}$, p^- can be chosen to be generated by $\gamma_2 \in p^-$ with $q(\gamma_2) = (-a)/p + \mathbb{Z}$. Analogously to before the subgroup of $p^+ \oplus p^-$ generated by (γ_1, γ_2) is isotropic of rank 1, and, since

$$p^{+n} = \bigoplus_{i=1}^{n/2} (p^+ \oplus p^-)$$

is also a possible decomposition of p^{+n}, this yields again an isotropic subgroup of the desired rank.

In total this shows that there exists a sublattice $K \subseteq L$ with $K \cong L(p)$.

Many natural subgroups of $\mathrm{O}(L)$ can be considered as discriminant kernels $\widetilde{\mathrm{SO}}^+(K)$ for sublattices $K \subseteq L$. The work of Gritsenko, Hulek and Sankaran in [GHS07a] on Hirzebruch-Mumford volumes gives a tool for the computation of these indices in some cases. If the sublattice is isomorphic to the rescaling of the underlying lattice, the computation is considerably simpler than the approach shown there.

Example 5.2.5. We present again a special case, slightly more general than the one before, the index of $\widetilde{\mathrm{O}}(L)$ in $\mathrm{O}(L)$ for $L = K(p)$, the rescaling of an unimodular rank-$2k$ lattice K by an odd prime p: Obviously

$$\left[\mathrm{O}(L) : \widetilde{\mathrm{O}}(L)\right] = |\operatorname{im}(\phi)|$$

for the natural map $\phi : \mathrm{O}(L) \to \mathrm{O}(\Delta_L)$. We note again that the discriminant form $\Delta_L = L'/L$ is isomorphic (as a finite group) to $(\mathbb{Z}/p\mathbb{Z})^{2k}$ or, more precisely

$$\Delta_L \cong p^{+\mathrm{rk}(L)}$$

and is equipped with the finite quadratic form $\overline{q}$, so $\mathrm{O}(\Delta_L)$ is the (finite) automorphism group of a (finite) quadratic form $\overline{q}$ over the finite field $\mathbb{F}_p$, whose order is well-known: By [CCN+85, Section 2.4] we have

$$|\mathrm{O}(\Delta_L)| = 2p^{k(k-1)} \left(p^k - 1\right) \prod_{i=1}^{k-1} \left(p^{2i} - 1\right).$$

If $\phi : \mathrm{O}(L) \to \mathrm{O}(\Delta_L)$ is surjective, this is the index of the discriminant kernel in $\mathrm{O}(L)$.

Unfortunately, the most used criterion for the surjectivity of ϕ cannot be used here: Nikulin in [Nik79, Theorem 1.14.2] states that $\phi : \mathrm{O}(L) \to \mathrm{O}(\Delta_L)$ is surjective if L splits a hyperbolic plane, which is never the case for a rescaling $L(p)$ of a unimodular lattice. However, the unpublished work of Miranda and Morrison [MM09] (attributing the case we are going to use again to Nikulin) proved:

Lemma 5.2.6 ([MM09, Corollary 7.8]). *Let L be an indefinite even lattice of rank $r = rk(L) \geq 3$ with discriminant form*

$$\Delta_L \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/d_r\mathbb{Z},$$

satisfying $d_1 \geq 1$ and $d_i | d_{i+1}$. If $d_2 = p^k$, $k > 0$, for some prime $p \equiv 3 \mod 4$, the lattice L is unique in its genus and the map

$$\phi : \mathrm{O}(L) \to \mathrm{O}(\Delta_L)$$

is surjective.

This enables us to state (noting $L(p)'/L(p) \cong (\mathbb{Z}/pZ)^{\mathrm{rk(L)}}$):

Corollary 5.2.7. *Let L be an indefinite even unimodular lattice of rank $r = 2k$ at least 3 and let $p \equiv 3 \mod 4$ be a prime, then the map $\phi : \mathrm{O}(L(p)) \to \mathrm{O}(\Delta_{L(p)})$ is surjective. $L(p)$ is unique in its genus and the index of the discriminant kernel $\widetilde{\mathrm{O}}(L(p))$ in $\mathrm{O}(L) \cong \mathrm{O}(L(p))$ is*

$$\left[\mathrm{O}(L) : \widetilde{\mathrm{O}}(L(p))\right] = 2p^{k(k-1)} \left(p^k - 1\right) \prod_{i=1}^{k-1} \left(p^{2i} - 1\right).$$

Since $\mathrm{SO}^+(L)$ has index 4 in $\mathrm{O}(L)$ we note

$$\left[\mathrm{O}(L(p)) : \widetilde{\mathrm{SO}}^+(L(p))\right] = 2^l \left[\mathrm{O}(L(p)) : \widetilde{\mathrm{O}}(L(p))\right]$$

for some $l \in \{0, 1, 2\}$.
The discriminant kernels are subgroups of the group $\mathrm{O}(q)$, whose properties can be generalized to arbitrary linear algebraic groups over $\mathbb{Q}$:

Definition 5.2.8. Let G be a linear algebraic group defined over $\mathbb{Q}$ and let $H \subseteq G$ be a subgroup of G. The subgroup H is called *arithmetic* if there exists a faithful representation $\rho : G \to \mathrm{GL}$ defined over $\mathbb{Q}$, such that $\rho(H)$ is commensurable with $\rho(G) \cap \mathrm{GL}(n, \mathbb{Z})$, i.e. $\rho(H) \cap \mathrm{GL}(n, \mathbb{Z})$ has finite index in both $\rho(H)$ and $\rho(G) \cap \mathrm{GL}(n, \mathbb{Z})$.

Obviously, the discriminant kernels are arithmetic subgroups of $\mathrm{O}(q)$. This is exactly what we meant by *arithmetic* in the chapter about general locally symmetric spaces. Such a group is always discrete and will act properly discontinuously on symmetric spaces of orthogonal type, which we will define in the next chapter.

Definition 5.2.9. A subgroup $H \subseteq \widetilde{\mathrm{O}}(L)$ of the form $\widetilde{\mathrm{SO}}^+(L(N))$ is called the *N-th principal congruence subgroup* of $\widetilde{\mathrm{O}}(L)$. Any subgroup of $\widetilde{\mathrm{O}}(L)$ containing a principal N-th congruence subgroup is called *congruence subgroup*.

For geometric reasons we will restrict ourselves at some point to the class of neat subgroups as in definition 1.1.10. Fortunately, this is no heavy restriction by the following result of Borel:

Proposition 5.2.10 ([Bor69])**.** *An arithmetic group has a neat subgroup of finite index.*

The proof given in [Bor69] is constructive in the following sense:

Lemma 5.2.11. *Let $\Gamma \subseteq \mathrm{GL}_n(\mathbb{Z})$ for some $n \in \mathbb{Z}, n > 0$, be an arithmetic subgroup. Denote by $\Phi_k(X)$ the k-th cyclotomic polynomial and define*

$$S_n = \prod_{k=2}^{n} \Phi_k(1).$$

For any prime $p \nmid S_n$, the finite-index congruence subgroup

$$\Gamma(p) = \{g \in \Gamma \subseteq \mathrm{GL}_n(\mathbb{Z}) |\, g \equiv 1 \mod p\}$$

is a neat subgroup of Γ.

Note that the value of S_n is independent of the arithmetic group Γ.

Remark 5.2.12. The following table gives some values of S_n for some $n \in \mathbb{Z}, n > 0$ of interest, p_n denoting the smallest prime not dividing S_n:

n	S_n	p_n
2	2	3
3	6	5
4	12	5
5	60	7
6	60	7
7	420	11
8	840	11
10	2520	11
12	27720	13
20	232792560	23
28	80313433200	29

Obviously, a subgroup of a neat group is itself neat, so this yields a plethora of neat subgroups of the orthogonal group of an even lattice L:

Lemma 5.2.13. *Let L be a lattice of dimension n. For p a prime not dividing S_n, the discriminant kernel $\widetilde{\mathrm{SO}}^+(L(p))$ is a neat subgroup of $\mathrm{O}(L) = \mathrm{O}(L(p))$.*

This finishes our introductory treatment of lattices and their automorphism groups. The next chapter will relate these to the theory of (locally) symmetric spaces and construct a class of examples, the *orthogonal locally symmetric spaces.*

6. Orthogonal (locally) symmetric spaces

We turn to a special case of orthogonal (locally) symmetric spaces: (locally) symmetric spaces constructed from the automorphism groups of lattices.
This chapter will introduce and examine this concept. The first section will present several models for orthogonal symmetric spaces; the second section will describe the geometry of the corresponding orthogonal locally symmetric spaces and their Baily-Borel compactifications. The third and final section treats the Siegel domain realizations of orthogonal symmetric spaces in some detail.
From now on we only consider lattices of signature $(2, n)$ with $n > 2$. This has several effects: On the one hand, the condition $n > 2$ ensures that the quadratic form of the lattices always has non-trivial rational zeroes, which will be important for the structure of the boundary of the corresponding orthogonal symmetric space; on the other hand, the fact that there are exactly 2 positive eigenvalues corresponds to the existence of a complex structure on the same variety by proposition 1.1.5, so these spaces will be special cases of the one treated in the first part of this thesis.
Note that the same would be true for a lattice of signature $(n, 2)$. However, by substituting q by $-q$, we can, without any loss of generality, restrict ourselves to the case of signature $(2, n)$.

6.1. Symmetric space of orthogonal type

Let (L, q) be an even lattice of signature $(2, n)$ and consider the orthogonal group $\mathrm{SO}^+(q)$ of the associated quadratic space. This is the real locus of a semisimple linear algebraic group defined over $\mathbb{Q}$, so we are in the situation of chapter 1. We can consider the associated symmetric space as in proposition 1.1.2.

Definition 6.1.1. Let $p \in V$ be a point. The quotient

$$\mathcal{D}_L := \mathrm{O}^+(q)/\mathrm{Stab}_p(\mathrm{O}^+(q)) \cong \mathrm{O}(2, n)/\left(\mathrm{O}(2) \times \mathrm{O}(n)\right) \cong \mathrm{SO}^+(2, n)/\left(\mathrm{SO}(2) \times \mathrm{SO}(n)\right)$$

is an irreducible symmetric space. It is the *orthogonal symmetric spaces corresponding to L.*

Proof. The group $\mathrm{SO}^+(2, n)$ is a connected Lie group and the stabilizer of a point is of the form $\mathrm{SO}(2) \times \mathrm{SO}(n)$ and satisfies proposition 1.1.2, so the latter quotient is a symmetric space as stated. The remaining isomorphisms are easy to see. □

All of the theory of the first chapter 1 applies to this symmetric space. We note that its compact dual is $\check{\mathcal{D}}_L \cong \mathrm{SO}(2 + n)/\mathrm{SO}(2) \times \mathrm{SO}(n)$.

Since this is a fairly abstract description of the orthogonal symmetric space, we will provide some more models of this very object. These are better suited for our purposes and are the right set-up for the definition of orthogonal modular forms.

Grassmannian model

Consider the quadratic space $V = L \otimes \mathbb{R}$ with its quadratic form q. Let $\mathrm{Gr}^+(L) = \mathrm{Gr}^+(V)$ be the space of oriented 2-dimensional positive definite subspaces of V with respect to q, the *oriented Grassmannian of L*. This is a real analytic manifold of dimension n. We note Witt's theorem:

Proposition 6.1.2. *Let (V, q) be a non-degenerate real quadratic space. Any isometry of $U \to U'$ of subspaces $U, U' \subseteq V$ extends to an isometry of V.*

Obviously, $\mathrm{O}(2, n)$ operates on $\mathrm{Gr}^+(L)$. By Witt's theorem, this action is transitive and the stabilizer of a point is exactly given by $\mathrm{O}(2) \times \mathrm{O}(n)$. Hence

$$\mathcal{D}_L \cong \mathrm{Gr}^+(L).$$

While the construction of Grassmannians $\mathrm{Gr}^+(L)$ works for lattices or quadratic forms in any signature, the $(2, n)$-case exhibits a special feature: The resulting manifold carries a natural complex structure by proposition 1.1.5 since $\mathrm{SO}(2) \times \mathrm{SO}(n)$ contains the torus $\mathrm{SO}(2)$. This can be seen explicitly in the so-called *projective model*.

Projective model

Consider the projectivization $\mathbb{P}(V(\mathbb{C}))$ of the complexification $V(\mathbb{C}) = V \otimes \mathbb{C} = L \otimes \mathbb{C}$ of L. This is a projective space of complex dimension $n+1$ and hence a complex manifold. The space $V(\mathbb{C})$ carries the $\mathbb{C}$-linear extension of the bilinear form of L. We note that the bilinear form is not well-defined on $\mathbb{P}(V(\mathbb{C}))$, but the relations

$$([Z], [Z]) = 0 \text{ and } ([Z], \overline{[Z]}) > 0$$

are independent of the choice of the representative Z of $[Z] \in \mathbb{P}(V(\mathbb{C}))$. Therefore we can consider the zero quadric

$$\mathcal{N} := \{[Z] \in \mathbb{P}(V(\mathbb{C})) |\, ([Z], [Z]) = 0\}$$

and its open subset

$$\mathcal{K} := \left\{[Z] \in \mathcal{N} |\, ([Z], \overline{[Z]}) > 0\right\}.$$

This subspace is a complex manifold of dimension n with two connected components of which we choose exactly one and denote it by $\mathcal{K}^+$. There is a natural action of $\mathrm{O}^+(L)$ on both $V(\mathbb{C})$ and $\mathbb{P}(V(\mathbb{C}))$, preserving the sets $\mathcal{N}$, $\mathcal{K}$ and $\mathcal{K}^+$.

Again, this complex manifold is isomorphic to our symmetric space (in the Grassmannian model) by the following consideration: For a point $p \in \mathrm{Gr}^+(L)$ which is positive definite plane $P \subseteq V$ choose an orthogonal basis (X_P, Y_P) and scale it to satisfy

$$q(X_P) = q(Y_P) > 0;$$

now consider the point $[Z_P] := [X_P + iY_P] \in \mathbb{P}(V(\mathbb{C}))$: It is actually in $\mathcal{K}$ since

$$(Z_P, Z_P) = q(X_P) - q(Y_P) = 0 \text{ and } \left(Z_P, \overline{Z_P}\right) = 2q(X_P) > 0.$$

Moreover we can assume that $[Z_P]$ is in $\mathcal{K}^+$ by simply switching X_P and Y_P if necessary. Starting with a point $[Z] = [X + iY] \in \mathbb{P}(V(\mathbb{C}))$ with $X, Y \in V$, the same computations show that this implies $(X, Y) = 0$ and $q(X) = q(Y) > 0$, so the plane spanned by X and Y is positive definite and defines a point in $\mathrm{Gr}^+(L)$. One can also check that these constructions are inverse to each other and equip the Grassmannian resp. the symmetric space $\mathcal{D}$ with a complex structure, making it a Hermitian symmetric space as in definition 1.1.1.

This is the *projective model.* It will be the model best suited for our first definition of orthogonal modular forms; moreover it shows that the orthogonal symmetric space has the structure of an algebraic variety, so we can apply algebraic as well as geometric methods to it.

Tube domain model

We turn to the third and final model of the orthogonal symmetric space. This follows the treatment in [Bru02].

To construct the so-called tube domain realization of $\mathcal{D}$ let us return to the underlying lattice L. Let $z \in L$ be a primitive isotropic and choose $z' \in L'$ with $(z, z') = 1$. The set

$$K = L \cap z^\perp \cap \left(z'\right)^\perp$$

is a Lorentzian sublattice (i.e. of signature $(1, n-1)$) of L. Now define the *tube domain*

$$\mathbb{H}_L = \{Z_K \in K \otimes \mathbb{C} \,|\, q_K(\mathrm{Im}(Z_K)) > 0\}$$

and choose one of its connected components. This complex manifold derives its name from the fact that for each real coordinate the allowed imaginary coordinate is restricted to a tube induced by the quadratic form of K. It is a realization as a Siegel domain of the first kind as in proposition 1.1.19.

This structure is again an incarnation of our symmetric domain $\mathcal{D}$: One can easily compute that for any $Z_K \in \mathbb{H}_L$, the point

$$[Z_L] = [(-q_K(Z_K) - q_L(z'))\, z + z' + Z_K] = [(-q_K(Z_K) - q_L(z'), 1, Z_K)]$$

is in $\mathcal{K}^+$ with the coefficient of z achieving the overall isotropy whilst the condition on the imaginary part ensures $(Z_L, \overline{Z_L}) > 0$; for the inverse map we note that any representative

$Z = X + iY$ of a $[Z] \in \mathcal{K}^+$ is not orthogonal to z': If it were, the real vectors X_K, Y_K would span a positive definite plane in the signature $(1, n-1)$-space $K \otimes \mathbb{R}$. Hence we can normalize $[Z] \in \mathcal{K}^+$ to be of the form $[(Z_K, 1, b)]$ for some $Z_K \in K \otimes \mathbb{C}$ and $b \in \mathbb{C}$. The same computations as before show that b is $-q_K(Z_k) - q_L(z')$ and the imaginary part of Z_K lies in the cone $q_K > 0$, so the map $[(Z_K, 1, b)] \mapsto Z_K \in \mathbb{H}_L$ is well-defined; moreover, the maps are biholomorphic.
Note that this realization depends on the choice of some primitive isotropic vector $z \in L$. Obviously, this is a concrete realization of the symmetric space $\mathcal{D}_L$ as a Siegel domain of the third kind (resp. even of the first kind) with respect to the rational boundary component $\mathcal{F}$ defined by $z \in L$.
Another remark: Scaling the lattice L yields another lattice $L(N)$ of the same signature, so the preceding theory gives a symmetric space $\mathcal{D}_{L(N)}$. This can be naturally identified with $\mathcal{D}$ by noticing

$$L(N) \otimes \mathbb{R} \cong \mathbb{R}^{n+2} \cong L \otimes \mathbb{R}.$$

Similarly, one can identify $\mathcal{D}_{L'} \cong \mathcal{D}_L$ for any sublattice $L' \subseteq L$ of the same rank by identifying $L' \otimes \mathbb{R} \cong L \otimes \mathbb{R}$.
If, by accident, the rescaling $L(N)$ is isomorphic to some sublattice of L, these two identifications are different: Fix a basis $\lambda_1, \ldots, \lambda_{n+2}$ of L. The first of these embeddings gives $L(N)$ in $L \otimes \mathbb{R}$ as the $\mathbb{Z}$-span of $\sqrt{N}\lambda_1, \ldots, \sqrt{N}\lambda_{n+2}$, so, in general, the images of vectors in $L(N)$ do not lie in $L \hookrightarrow L \otimes \mathbb{R}$. This contrasts the second embedding in which obviously $L(N) \cong L' \subseteq L \hookrightarrow L \otimes \mathbb{R}$. One can switch between these two identifications by the obvious base change $L' \cong L(N)$.
The next section will associate a canonical locally symmetric space to every lattice L.

6.2. Locally symmetric spaces of orthogonal type

Given an arithmetic subgroup Γ of $\mathrm{SO}^+(2, n)$ we can construct locally symmetric spaces $\Gamma \backslash \mathcal{D}$. In the case of $\mathcal{D} = \mathcal{D}_L$ for a lattice L there is a canonical choice: the discriminant kernel

$$\widetilde{\mathrm{SO}}^+(L) = \ker\left(\mathrm{SO}^+(L) \to \mathrm{O}(\Delta_L)\right).$$

Definition 6.2.1. Let L be an even lattice of signature $(2, n)$ for some $n > 2$. We define

$$X(L) = \widetilde{\mathrm{SO}}^+(L) \backslash \mathcal{D}_L \cong \widetilde{\mathrm{SO}}^+(L) \backslash \mathrm{SO}^+(2, n) / \left(\mathrm{SO}(2) \times \mathrm{SO}(n)\right)$$

and call it the *locally symmetric space associated to* L or a *orthogonal locally symmetric space* without a reference to the inducing lattice.

Note that this locally symmetric space has finite-quotient singularities in general. By the theory of the preceding section, this is a smooth variety and complex manifold if Γ is torsion-free. One can find a smooth cover for any $X(L)$ by working with a suitable rescaling $L(N)$ to get a neat discriminant kernel, and the identification $\mathcal{D}_L \cong \mathcal{D}_{L(N)}$. We want to relate this to the theory of Shimura varieties as in section 1.2.

Interpretation as Shimura varieties

We can understand this locally symmetric space in the context of Shimura varieties, too, and follow the exposition in [AGHMP17].
Let (V, q) be non-degenerate quadratic space coming from a lattice L and consider its *Clifford algebra*

$$C(V) = \left(\bigoplus_{k=0}^{\infty} V^{\otimes k} \right) \Big/ \langle x \otimes x - q(x) \rangle_{x \in V},$$

the quotient of the tensor algebra by the relations $x \otimes x - q(x)$. There is a natural $\mathbb{Z}/2\mathbb{Z}$-grading on $C(V)$, so

$$C(V) = C^+(V) \oplus C^-(V).$$

This defines the *general spin group* by

$$\mathrm{GSpin}(V) = \left\{ g \in C^+(V)^* \,|\, gVg^{-1} = V \right\}.$$

This definition can be generalized to yield an algebraic group scheme $\mathrm{GSpin}(V)$ over $\mathbb{Q}$. Analogously, one can generalize the notion of Clifford algebras to quadratic spaces over fields F coming from R-lattices L over $R \subseteq F$. We note that any R-lattice L induces an R-order $C(L)$ in $C(V)$, so one can define $\mathrm{GSpin}(L)$ analogously to the case of quadratic spaces and we have

$$\mathrm{GSpin}(L) \subseteq \mathrm{GSpin}(V).$$

Let $\mathcal{D} = \mathcal{D}_L$ be the symmetric space of L and consider its Grassmannian model. For any two-dimensional positive definite plane $Z = \mathbb{R}X + \mathbb{R}Y \in \mathrm{Gr}^+(L)$ there is the canonical inclusion $C(Z) \hookrightarrow C(V \otimes \mathbb{R})$ which can be extended to

$$\mathbb{C} \hookrightarrow C(Z) \hookrightarrow C(V \otimes \mathbb{R})$$

by

$$a + ib \mapsto aX + \frac{xy}{\sqrt{q(x)q(y)}} Y.$$

The restriction to

$$\mathbb{S}(\mathbb{R}) = \left(\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \right)(\mathbb{R}) \cong \mathbb{C}^* \hookrightarrow C(V \otimes \mathbb{R})^*$$

gives a map $\alpha_Z : \mathbb{S}(\mathbb{R}) \to \mathrm{GSpin}(V)(\mathbb{R})$. The following well-known result lifts this to a morphism of real algebraic groups:

Lemma 6.2.2 ([AD15, Lemma 3.2.2]). *Every representation of* $\mathrm{U}_1(\mathbb{R}) \cong \mathbb{C}^*$ *as a real Lie group arises from an algebraic representation. If the representation is real, then it arises as an algebraic representation of* $\mathbb{R}$*-groups.*

This construction identifies $\mathcal{D}$ with a $G(\mathbb{R})$-conjugacy class of morphisms

$$\mathbb{S} \to \mathrm{GSpin}(V)_{\mathbb{R}}$$

and one can show that the conditions for a Shimura datum are satisfied as well.
It remains to explain the connection to locally symmetric spaces of orthogonal type: Consider the profinite completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$ and $\widehat{L} = L_{\widehat{\mathbb{Z}}}$ and define

$$\mathcal{U} = \mathrm{GSpin}(V)(\mathbb{A}_f) \cap C\left(\widehat{L}\right)^*,$$

then the Shimura variety

$$\begin{aligned} X_{\mathcal{U}} &= \mathrm{GSpin}(V)(\mathbb{Q})\backslash(\mathcal{D}_L \times \mathrm{GSpin}(V)(\mathbb{A}_f))/\mathcal{U} \\ &= \mathrm{GSpin}(V)(\mathbb{Q})\backslash\left(\mathcal{D}_L \times \mathrm{GSpin}(V)(\mathbb{A}_f)\right)/\mathrm{GSpin}(V)(\mathbb{A}_f) \cap C\left(\widehat{L}\right)^* \end{aligned}$$

is exactly the locally symmetric space

$$X(L) = \widetilde{\mathrm{SO}}^+(L)\backslash\mathcal{D}_L$$

corresponding to L.
This interpretation of orthogonal locally symmetric spaces as Shimura varieties allows us to use very general results of the theory of the latter during our future work on locally symmetric spaces $X(L)$. Besides that, we will use this interpretation as a source for natural vocabulary on these varieties. The interested reader may consult [AGHMP17, Section 2] for a short survey or [Hör10] and [AD15] for more details on this point of view.
In particular, for a fixed L, we will use the phrases 'orthogonal symmetric space' and 'orthogonal Shimura variety' interchangeably for $X(L)$.

Boundary components of orthogonal locally symmetric spaces

We return to more general locally symmetric spaces of orthogonal type. For L as before and arithmetic $\Gamma \subseteq \mathrm{O}(L)$, the rationality of boundary components in the construction of the Baily-Borel compactification of $\Gamma\backslash\mathcal{D}$ can be related to the structure of L.
Consider the closure $\overline{\mathcal{D}_L}$ of $\mathcal{D}_L$ in the projective model (this is equivalent to doing this in $\mathfrak{p}_+$ via the Harish-Chandra embedding): The boundary is then given by

$$\partial\overline{\mathcal{D}_L} = \left\{[Z] \in \mathcal{N} \,\middle|\, \left([Z], \overline{[Z]}\right) = 0\right\},$$

that is, isotropic subspaces of the quadratic space (V, q). The notion of rationality for these boundary components reduces to the condition that the generators of these isotropic subspaces are in L, so:

Proposition 6.2.3. *The k-dimensional rational boundary components of $\mathcal{D}_L$ are given by completely isotropic $(k+1)$-dimensional sublattices of L. In the case of signature $(2, n)$ there are exactly two possibilities for this. A rational boundary component $\mathcal{F}$ of $\mathcal{D}_L$ is given by one of the following:*

i) F is zero-dimensional, corresponding to an isotropic line spanned by some primitive isotropic vector $e \in L$

ii) F is one-dimensional, corresponding to an isotropic plane, spanned by two linearly independent primitive isotropic vectors $e, f \in L$.

This can be found, for example, in [BF01, Remark 2.1].
By construction of the Baily-Borel compactification any of the rational boundary components $\mathcal{F} \subseteq \overline{\mathcal{D}_L}$ defines a boundary component F of $\overline{\Gamma\backslash\mathcal{D}_L}$ by projection modulo Γ.
In our setting, we usually speak of them as zero- resp. one- dimensional *cusps*. They are of the following form:

Proposition 6.2.4. *A cusp of $\overline{\Gamma\backslash\mathcal{D}_L}^{BB}$ is of one of the following two types:*

i) A zero-dimensional cusp F_0 of $\overline{\Gamma\backslash\mathcal{D}_L}^{BB}$ is just a point.

ii) A one-dimensional cusp F_1 of $\overline{\Gamma\backslash\mathcal{D}_L}^{BB}$ is isomorphic to the modular curve

$$\Gamma_1\backslash\mathbb{H}$$

with Γ_1 the group of automorphisms of the defining isotropic plane which are induced by Γ acting on $\mathcal{F} \subset \mathcal{D}_L$. Alternatively, this can be described as the integral version of the component $G_h(\mathcal{F}_1)$ of the Levi decomposition of the parabolic $\mathcal{P}(\mathcal{F}_1)$, cf. theorem 1.1.16, for $\mathcal{F}_1$ any preimage of F_1.

Number of cusps

The number of cusps of a Baily-Borel compactification is a geometric property of $\Gamma\backslash\mathcal{D}$. In general, it is a hard problem to determine the number of cusps of a locally symmetric space $\Gamma\backslash\mathcal{D}_L$. By the preceding consideration, the number of k-dimensional cusps is equal to the number of Γ-orbits of $k+1$-dimensional isotropic sublattices $I \subseteq L$. There are a few examples in which these are known:

Example 6.2.5. By the uniqueness of indefinite unimodular lattices and the classification of the unimodular definite lattices in proposition 5.1.8 we see:

- Let $L = II_{2,10}$ be the unique unimodular lattice of signature $(2, 10)$, then $X(L)$ has exactly one zero-dimensional cusp corresponding to the lattice $II_{1,9}$ in a decomposition $II_{2,10} = II_{1,1} \oplus II_{1,9}$. Similarly, there is only one one-dimensional cusp corresponding to $\mathrm{E}_8(-1)$ in a decomposition $II_{2,10} = II_{1,1} \oplus II_{1,1} \oplus \mathrm{E}_8(-1)$.
- Let $L = II_{2,26}$ be the unique unimodular lattice of signature $(2, 26)$, then $X(L)$ has again exactly one zero-dimensional cusp corresponding to the lattice $II_{1,25}$ in a decomposition $II_{2,26} = II_{1,1} \oplus II_{1,25}$. There are 24 one-dimensional cusps corresponding to the 24 Niemeier lattices.

Generally, for an even unimodular lattice of signature $(2, 8k+2)$ the number of zero-dimensional cusps is 1 (this is just the uniqueness of even unimodular indefinite lattices) and the number of one-dimensional cusps is equal to the number of lattices in the genus $I\!I_{0,8k}$. This works by virtue of the Eichler criterion as in lemma 5.2.2.

We consider now the the case of a rescaling $L = L_0(N)$ of an even unimodular lattice L_0 of signature $(2, n)$ with $\Gamma = \widetilde{\mathrm{SO}}^+(L)$ its discriminant kernel. These lattices do not satisfy the Eichler criterion, but it is possible to get further explicit results in some additional cases, cf. [AD14] and in particular [AD15]. Fortunately the case of most interest to us is treated there implicitly: The results carry over with the same proofs.

Proposition 6.2.6 ([AD15, Theorem 5.4.2])**.** *Let $M = M_0(N)$ the rescaling by $N > 1$ of a unimodular lattice M_0 of the form $M_0 = I\!I_{1,1} \oplus L_0 = I\!I_{1,1} \oplus I\!I_{1,1} \oplus K_0$ with signature $(2, n)$. Let*

$$\Gamma_{M_0(N)} = \widetilde{\mathrm{SO}}(M_0(N)) \text{ resp. } \Gamma_{L_0(N)} = \widetilde{\mathrm{SO}}(L_0(N))$$

be the discriminant kernels of the (scaled) lattices and assume that $M \subseteq M_0$. Denote by $\overline{X}^{BB}$ the Baily-Borel compactification of $\Gamma_{M_0(N)} \backslash \mathrm{SO}^+(2,n)/\mathrm{SO}^+(2) \times \mathrm{SO}^+(n)$. Assume moreover that $\Gamma_{M_0(N)}$ is neat, e.g. by choosing N large enough, cf. remark 5.2.12.

i) The number of zero-dimensional cusps of $\overline{X}^{BB}$ is

$$\frac{\left[\Gamma_{M_0} : \Gamma_{M_0(N)}\right]}{2N^n \left[\Gamma_{L_0} : \Gamma_{L_0(N)}\right]}.$$

ii) The number of one-dimensional cusps of $\overline{X}^{BB}$ is

$$\frac{2\left[\Gamma_{M_0} : \Gamma_{M_0(N)}\right]}{N^{2n} \prod_{p|N} (1 - p^{-2})} \cdot \sum_{K \in \mathrm{gen}(K_0)} \frac{1}{|\mathrm{O}(K)|}.$$

Note that the number of lattices in a given genus is finite, moreover $|O(K)|$ is finite as well, since any appearing K is a negative definite lattice.

We apply this to the standard example $X(I\!I_{2,10}(p))$ for $p \equiv 3 \mod 4$:

Example 6.2.7. The number of zero-dimensional cusps of the Baily-Borel compactification of $I\!I_{2,10}(p)$ is

$$\frac{\left[\mathrm{SO}(I\!I_{2,10}) : \widetilde{\mathrm{SO}}(I\!I_{2,10}(p))\right]}{2p^{10}\left[\mathrm{SO}(I\!I_{1,9}) : \widetilde{\mathrm{SO}}(I\!I_{1,9}(p))\right]} = \frac{(p^6-1)(p^5+1)}{2}.$$

The number of one-dimensional cusps is

$$\frac{2\left[\mathrm{SO}(I\!I_{2,10}) : \widetilde{\mathrm{SO}}(I\!I_{2,10}(p))\right]}{p^{20}\,(1-p^{-2})} = \frac{2^l p^{12}(p^4-1)(p^6-1)^2(p^8-1)(p^{10}-1)}{348364800}$$

for some $l \in \{0, 1, 2\}$, since $|\mathrm{O}(\mathrm{E}_8)| = 4!6!8! = 696729600 = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$.
These indices can be computed by the surjectivity of the map $\mathrm{O}(L) \to \Delta_{\mathrm{L}}$ via example 5.2.5.

We can describe the one-dimensional cusps of $\widetilde{\mathrm{SO}}^+(L)\backslash\mathcal{D}_L$ even more explicitly for $L = L_0(N)$: The group Γ_1 as in proposition 6.2.4 has the simpler description as the N-th principal congruence subgroup

$$\Gamma(N) := \{M \in \mathrm{SL}_2(\mathbb{Z}) | M \equiv \mathrm{id} \mod N\}$$

of $\mathrm{SL}_2(\mathbb{Z})$ as we saw in example 5.2.3.
In particular: Any one-dimensional cusp F_1 is isomorphic to the modular curve $\Gamma(N)\backslash\mathbb{H}$.

6.3. Siegel domain realizations

The realization of symmetric spaces as Siegel domains of the third kind is of great importance for our later tasks; moreover, it serves well as an opportunity to give concrete descriptions of the decomposition of parabolic subgroups for orthogonal symmetric spaces. For some of the upcoming considerations it will be convenient to work with a fixed choice of coordinates.
Let (L, q) be a fixed even lattice of signature $(2, n)$ for $n > 4$ with quadratic form q and induced bilinear form denoted by $(\cdot, \cdot)$ of level N and let $\Gamma \subseteq \mathrm{SO}^+(L)$ be a fixed arithmetic subgroup. We assume further, that any primitive isotropic vector $l \in L$ satisfies $(l, L) = N\mathbb{Z}$.
As can be seen, for example in [Bru14, Lemma 5.1 and section 6], these properties imply

$$L = I\!I_{1,1}(N) \oplus I\!I_{1,1}(N) \oplus \Lambda$$

with a negative definite lattice Λ. For the sake of completeness we give the construction of this decomposition below: The fact that $n > 4$ and Meyer's theorem implies the existence of (primitive) isotropic vectors in L. Choose any primitive isotropic $e \in L$. Since q is non-degenerate, the orthogonal complement $e^\perp$ of e is not the whole of L and hence we can find an $e' \in L'$ with $(e, e') = 1$. Substituting e' with $e'' = Nq(e') + e'$ allows us to assume $e' \in L$ with $(e, e') = N$ and $q(e') = 0$. The sublattice spanned by e and e' is isomorphic to the unique (hyperbolic) lattice $I\!I_{1,1}(N)$ in the genus $I\!I_{1,1}(N)$ and $K = L \cap e^\perp \cap e'^\perp$ is again an even lattice of level N with $(k, K) = N\mathbb{Z}$ for every primitive isotropic $k \in K$, but this time of signature $(1, n-1)$.
The existence of isotropic vectors in L is still guaranteed by Meyer's theorem. Analogously to before we get $f, f' \in K$ and

$$\Lambda = L \cap f^\perp \cap (f')^\perp$$

such that

$$L = K \oplus I\!I_{1,1}(N) = \Lambda \oplus I\!I_{1,1}(N) \oplus I\!I_{1,1}(N)$$

with Λ now being a negative definite lattice of signature $(0, n-2)$. We can assume that, for any primitive isotropic vector $e \in L$ we have chosen a basis of L such that the quadratic form is induced by a matrix of the form

$$A = \begin{pmatrix} 0 & N & 0 \\ N & 0 & 0 \\ 0 & 0 & A_0 \end{pmatrix}$$

where the entries of the last row/column are in fact $(n \times 1)$ - resp. $(1 \times n)$ - block matrices and A_0 is some Gram matrix of the lattice K. In this case the basis is given by the ordered tuple $(e, e', \ldots)$, where $\ldots$ denotes some basis of K.
Moreover, for any two mutually orthogonal primitive isotropic vectors $e, f \in L$, we can choose a basis $(e, f, e', f', \ldots)$, where $\ldots$ denotes some basis for the definite lattice Λ, such that we can assume A to be of the form

$$\begin{pmatrix} 0 & N\,\mathrm{id} & 0 \\ N\,\mathrm{id} & 0 & 0 \\ 0 & 0 & A_1 \end{pmatrix},$$

where now the four upper left entries are (2×2) - block matrices, A_1 is a Gram matrix of the lattice Λ and the remaining entries are block matrices of suitable dimensions.
We consider now the symmetric space $\mathcal{D} = \mathcal{D}_L$.
Let $\mathcal{F}_0$ be a zero-dimensional rational boundary component of $\mathcal{D}$, $\mathcal{P}(\mathcal{F}_0)$ its associated parabolic stabilizer in $\mathrm{SO}^+(L \otimes \mathbb{R})$ and let $e \in L$ be the primitive isotropic vector corresponding to $\mathcal{F}_0$. We choose a basis of the form $(e, e', \ldots)$ of the lattice L as just described. By the general theorem 1.1.16 we know that

$$\mathcal{P}(\mathcal{F}_0) = (\mathcal{U}(\mathcal{F}_0) \rtimes \mathcal{V}(\mathcal{F}_0)) \rtimes (\mathcal{M}(\mathcal{F}_0) \cdot G_h(\mathcal{F}_0) \cdot G_l(\mathcal{F}_0)),$$

but now we are able to determine the factors more explicitly in this basis, cf. [Fio13, Section 5.4]. A missing entry means that it is equal to zero, an entry $*$ means that this entry is determined by the remaining one and the fact that the overall matrix is an element of $\mathrm{SO}^+(2, n)$.

$$\mathcal{P}(\mathcal{F}_0) \cong \mathbb{R}^n \rtimes \left(\mathrm{SO}^+(L \otimes \mathbb{R}) \cdot \mathbb{R}^*\right)$$

with

$$\mathcal{U}(\mathcal{F}_0) \cong \mathbb{R}^n = \begin{pmatrix} 1 & * & * \\ & 1 & \\ & \overrightarrow{d} & \mathrm{id} \end{pmatrix},$$

for $\overrightarrow{d} \in \mathbb{R}^n$,

$$G_h(\mathcal{F}_0) \cong \mathbb{R}^* = \begin{pmatrix} t & & \\ & t^{-1} & \\ & & \mathrm{id} \end{pmatrix},$$

for $t \in \mathbb{R}^*$,

$$G_l(\mathcal{F}_0) \cong \mathrm{SO}^+(K \otimes \mathbb{R}) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & T \end{pmatrix},$$

for $T \in \mathrm{SO}^+(K \otimes \mathbb{R})$ and $\mathcal{V}(\mathcal{F}_0), \mathcal{M}(\mathcal{F}_0)$ trivial.
This can be computed by explicitly determining the elements of $\mathrm{SO}(2, n)$ fixing the ray through e as well its unipotent radical and its center. The unipotent radical can be characterized in a computationally useful way by [Gar97].
Let now $\mathcal{F}_1$ be a one-dimensional rational boundary component of $\mathcal{D}$, $\mathcal{P}(\mathcal{F}_1)$ its associated parabolic stabilizer in $\mathrm{SO}^+(L \otimes \mathbb{R})$ and $e, f \in L$ be the primitive isotropic vectors corresponding to $\mathcal{F}_1$ by proposition 6.2.3. We choose a basis of the form $(e, e', f, f' \ldots)$ of the lattice L analogously to before. Then

$$\mathcal{P}(\mathcal{F}_1) \cong \mathcal{W}(\mathcal{F}_1) \rtimes \left(\mathrm{SO}^+(\Lambda \otimes \mathbb{R}) \cdot \mathrm{SL}_2(\mathbb{R}) \cdot \mathbb{R}^*\right)$$

with the unipotent radical

$$\mathcal{W}(\mathcal{F}_1) = \begin{pmatrix} \mathrm{id} & a_2(u) & * \\ & \mathrm{id} & \\ & \overrightarrow{d} & \mathrm{id} \end{pmatrix},$$

for $\left(\overrightarrow{d}, u\right) \in \left(\mathbb{R}^{n-2}\right)^2 \times \mathbb{R}$ and $a_2(u) \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ depending on u.
The group $\mathcal{W}(\mathcal{F}_1)$ is the *Heisenberg group* $H(L \otimes \mathbb{R}) \cong \left(\mathbb{R}^{n-2} \times \mathbb{R}^{n-2}\right) \ltimes \mathbb{R}$ with product

$$[x_1, y_1, r_1] \cdot [x_2, y_2, r_2] = \left[x_1 + x_2, y_1 + y_2, r_1 + r_2 + \frac{1}{2}((x_1, y_2) - (x_2, y_1))\right].$$

Its center is

$$\mathcal{U}(\mathcal{F}_1) = \begin{pmatrix} \mathrm{id} & a(u) & * \\ & & \\ & \overrightarrow{d} & \mathrm{id} \end{pmatrix},$$

for $u \in \mathbb{R}$ and $a(u) = \left(\begin{smallmatrix} 0 & u \\ -u & 0 \end{smallmatrix}\right) \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$; one has

$$\mathcal{V}(\mathcal{F}_1) = \mathcal{W}(\mathcal{F}_1)/\mathcal{U}(\mathcal{F}_1) = \begin{pmatrix} \mathrm{id} & & * \\ & \mathrm{id} & \\ & \overrightarrow{d} & \mathrm{id} \end{pmatrix}$$

for $\overrightarrow{d} \in (\mathbb{R}^{n-2})^2$. This has a complex structure via $\mathcal{V}(\mathcal{F}_1) \cong (\mathbb{R}^{n-2})^2 \cong \mathbb{C}^{n-2}$.
Moreover:

$$\mathcal{M}(\mathcal{F}_1) \cong \mathrm{SO}^+(\Lambda \otimes \mathbb{R}) = \begin{pmatrix} \mathrm{id} & & \\ & \mathrm{id} & \\ & & T \end{pmatrix},$$

for $T \in \mathrm{SO}^+(\Lambda \otimes \mathbb{R})$,

$$G_h(\mathcal{F}_1) \cong \mathrm{SL}_2(\mathbb{R}) = \begin{pmatrix} a & & \\ & a^{-T} & \\ & & \mathrm{id} \end{pmatrix},$$

for $a \in \mathrm{SL}_2(\mathbb{R})$ and

$$G_l(\mathcal{F}_1) \cong \mathbb{R}^* = \begin{pmatrix} t \cdot \mathrm{id} & & \\ & t^{-1} \cdot \mathrm{id} & \\ & & \mathrm{id} \end{pmatrix}$$

for $t \in \mathbb{R}^*$.

We can use this to describe the cones $\mathcal{C}(\mathcal{F}_0)$ and $\mathcal{C}(\mathcal{F}_1)$; recall the characterization

$$\mathcal{C}(\mathcal{F}) \cong G_l(\mathcal{F}) / \left(G_l(\mathcal{F}) \cap K\right).$$

This yields

$$\mathcal{C}(\mathcal{F}_0) = \{\overrightarrow{u} \in \mathcal{U}(\mathcal{F}_0) \cong K \otimes \mathbb{R} \cong \mathbb{R}^n |\, q_K(\overrightarrow{u}) > 0, u_1 > 0\}$$

where we assume that the basis for K is such that the quadratic form $q_K = q_L|_K$ of K is given by

$$u_1^2 - \sum_{k=2}^{n} u_k^2.$$

In more physical parlance, this is the positive time-like cone with positive time coordinate in the Lorentzian space $(K \otimes_{\mathbb{Z}} \mathbb{R}, q_K)$ of signature $(1, n-1)$. The symmetry group $G_l(\mathcal{F}_0)$ can be identified with the group of autochronous symmetries of the Lorentzian space. There is also an interpretation as a model for hyperbolic $n-1$-space. For all of this, see [Con83].

The situation is simpler for a one-dimensional rational boundary component $\mathcal{F}_1$: The cone $\mathcal{C}(\mathcal{F}_1)$ is

$$\mathcal{C}(\mathcal{F}_1) = \{u \in \mathcal{U}(\mathcal{F}_1) \cong \mathbb{R} |\, u > 0\} \cong \mathbb{R}_{>0}.$$

With this information we can realize $\mathcal{D}_L$ as Siegel domains of the third kind in the following explicit way:

Proposition 6.3.1. *The realization as a Siegel domain of the third kind with respect to a zero-dimensional boundary component $\mathcal{F}_0$ is*

$$\begin{aligned} \mathcal{D}_L &\cong \{x \in \mathbb{C}^n |\, \mathrm{Im}\, x \in \mathcal{C}(\mathcal{F}_0)\} \\ &\cong \{x \in \mathbb{C}^n |\, q_K(\mathrm{Im}\, x) > 0, \mathrm{Im}\, x_1 > 0\} \end{aligned}$$

with $\mathcal{U}(\mathcal{F}_0) \otimes \mathbb{C} \cong \mathbb{C}^n$ and trivial $\mathcal{V}(\mathcal{F}_0)$.

The realization as a Siegel domain of the third kind with respect to a one-dimensional boundary component $\mathcal{F}_1 \cong \mathbb{H}$ is

$$\mathcal{D}_L \cong \left\{(x, y, z) \in \mathbb{C} \times \mathbb{C}^{n-2} \times \mathbb{H} \,|\, \mathrm{Im}\, x + \frac{q_\Lambda(\mathrm{Im}\, y)}{\mathrm{Im}\, z} \in \mathbb{R}_{>0}\right\}$$

with $\mathcal{U}(\mathcal{F}_1) \otimes \mathbb{C} \cong \mathbb{C}$, $\mathcal{V}(\mathcal{F}_1) \cong \mathbb{C}^{n-2}$ and the cone $\mathcal{C}(\mathcal{F}_1) \cong \mathbb{R}_{>0}$.

If we choose the boundary components $\mathcal{F}_0$ and $\mathcal{F}_1$ compatible with each other, we can relate these two realizations. Let $\mathcal{F}_0 \subseteq \overline{\mathcal{F}_1}$, i.e. $\mathcal{F}_0$ is defined by an isotropic vector $e \in L$ and $\mathcal{F}_1$ by e and an additional isotropic $f \in L$.
Consider the parabolics in the basis (e, e', f, f'), that is, the first two entries in the descriptions of the factors of $\mathcal{P}(\mathcal{F}_0)$ correspond to the coordinates of f and f'. In these coordinates we can write

$$q_K((x_1, x_2, \overrightarrow{x})) = x_1 x_2 + q_\Lambda(\overrightarrow{x}).$$

Identifying the x_1-coordinate in the realization with respect to $\mathcal{F}_0$ with the z-coordinate in the realization with respect to $\mathcal{F}_1$ and the x_2-coordinate with the x-coordinate in the respective realizations shows that these indeed describe the same space $\mathcal{D}_L$.
As a final remark, we want to clarify the relation of the objects of the preceding section in the light of the identification

$$\mathcal{D}_L \cong \mathcal{D}_{L(N)}$$

for $L(N)$ a rescaling of L. This identification is compatible with all the constructions above: We embed $L(N)$ as $\sqrt{N}L \hookrightarrow L \otimes \mathbb{R}$, so we get a bijection $\sqrt{N}\cdot : L \otimes \mathbb{R} \to L \otimes \mathbb{R}$ between the primitive isotropic vectors in L and $L(N)$ inside $L \otimes \mathbb{R}$. The choice of bases as in section 6.3 is compatible as well.
As subspaces of $L \otimes \mathbb{R}$, the rational boundary components of $\mathcal{D}_L$ and $\mathcal{D}_{L(N)}$ are identical, even though the rationality comes from generators in L in the first case and their $\cdot\sqrt{N}$-images in the second. Analogously, this identification yields

$$\mathcal{U}(\mathcal{F}) \cong \mathcal{U}(\mathcal{F}')$$

and

$$\mathcal{C}(\mathcal{F}) \cong \mathcal{C}(\mathcal{F}'),$$

where $\mathcal{F}'$ is the boundary component of $\mathcal{D}_L$ corresponding to the boundary component $\mathcal{F}$ of $\mathcal{D}_{L(N)}$ under the identification. The internal rational structure as subobjects of $L \otimes \mathbb{R}$ differs by multiplication by $\sqrt{N}$.
This concludes our first encounter with (locally) symmetric spaces of orthogonal type. The next chapter will finally introduce the concept of *orthogonal modular forms* as highly symmetric $\mathbb{C}$-valued functions on $\mathcal{D}_L$.

7. Orthogonal modular forms

In the preceding chapter we introduced the notion of symmetric and locally symmetric spaces corresponding to even non-degenerate lattices of signature $(2, n)$.
In this chapter we will introduce $\mathbb{C}$-valued functions on these spaces. As the definition of locally symmetric spaces encodes a lot of symmetry into them, any function on a locally symmetric space is highly symmetric when considered on the original symmetric space. Theses function are called *modular forms* or *automorphic forms* and will be the main object of interest in this chapter.
The first section of this chapter will explain the general definition and give some (degenerate) examples; the second part will present a very powerful mechanism to construct a large class of these orthogonal modular forms.

7.1. Definition and examples

Let L be a fixed even non-degenerate lattice of signature $(2, n)$ with $n \geq 3$.
Working with this fixed lattice allows us to write $\mathcal{D} := \mathcal{D}_L$, $\mathbb{H} = \mathbb{H}_L$ and

$$X := X(L) = \widetilde{\mathrm{SO}^+(L)} \backslash \mathcal{D}_L.$$

The notion of orthogonal modular forms for the lattice L may be defined on the cone

$$\tilde{\mathcal{K}}^+ := \left\{ Z \in \mathcal{K}^+ \mid [Z] \in \mathcal{K}^+ \right\}$$

lying over $\mathcal{K}^+$ for

$$\mathcal{K} = \left\{ [Z] \in \mathbb{P}(L \otimes \mathbb{C}) \middle| \, ([Z], [Z]) = 0 \text{ and } ([Z], \overline{[Z]}) > 0 \right\}$$

corresponding to L as in the description of the projective model in section 6.1.

Definition 7.1.1. Let $k \in \mathbb{Z}$, $\Gamma \subseteq \mathrm{O}^+(L)$ be an arithmetic subgroup and χ a character of Γ. An *orthogonal modular form of weight k and character χ* is a meromorphic map $F : \tilde{\mathcal{K}}^+ \to \mathbb{C}$ satisfying

i) F is homogeneous of degree $-k$: $F(tZ) = t^{-k} F(Z)$ for every $t \in \mathbb{C}^*$

ii) F is (almost) invariant under Γ, i.e. $F(\gamma Z) = \chi(\gamma) F(Z)$ for all $\gamma \in \Gamma$

Orthogonal modular forms of given weight k, group Γ and character χ form a complex vector space, denoted by $M_k(\Gamma, \chi)$. If the character χ is trivial we omit it from the notation.

We will see in a moment that there are two more characterizations of modular forms, one of them more suited for analytical purposes, the other one for the use of methods from algebraic geometry.
Substituting *meromorphic* by *holomorphic* in the definitions above defines *holomorphic orthogonal modular forms.*
We recall that there is another realization of $\mathcal{D}$ as a bounded domain inside $\mathfrak{p}_-$, the $-i$-eigenspace of the complex structure of the complexification of Lie algebra $\mathfrak{g}$ of G, so it makes sense to speak about the closure of $\mathcal{D}$: The notion of *cusp forms* is defined in the obvious way as those modular forms vanishing on the boundary of $\mathcal{D}$ in this realization. The vector space of cusp forms of given weight for a group Γ is naturally denoted by $S_k(\Gamma)$.
By the so-called *Koecher principle* a holomorphic orthogonal modular form is bounded if $n \geq 3$, so they extend naturally to $\overline{\mathcal{D}}$. If $n \leq 2$ this boundedness condition has to be included into the definition of orthogonal modular forms to yield a well-behaved theory.
We can use the tube domain realization $\mathbb{H}$ of $\mathcal{D}$ to reformulate the notion of modular forms:

Definition 7.1.2. Let $k \in \mathbb{Z}$, $\Gamma \subseteq \mathrm{O}^+(L)$ be an arithmetic subgroup and χ a multiplier system of Γ. A *meromorphic modular form of weight k and multiplier system χ* is a meromorphic function $F : \mathbb{H} \to \mathbb{C}$ satisfying for all $\gamma \in \Gamma$

$$F(\gamma Z) = \chi(\gamma) j(\gamma, Z)^k F(Z)$$

with the automorphy factor $j(\gamma, Z) \in \mathbb{C}^*$ satisfying

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z) j(\gamma_2, z)$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and $z \in \mathbb{H}$.

This definition translates one-to-one to the one given for the projective model: The appearance of the factor of automorphy is due to transfer maps between the different models.
We give a final, geometric characterization for modular forms on the locally symmetric space $\Gamma\backslash\mathcal{D}$. This has the advantage of incorporating the transformation behavior of orthogonal modular forms into their well-definedness, essentially translating an arithmetic condition on an analytic object to a geometric condition on an arithmetic object.

Proposition 7.1.3 ([GHS07a, Section 1])**.** *Let $\Gamma \subseteq \mathrm{O}^+(L)$ be neat. Orthogonal modular forms for the trivial character and of weight* 1 *define a line bundle $\mathcal{L}$ on $X(\Gamma) = \Gamma\backslash\mathcal{D}$. Orthogonal modular forms of weight $n = \dim X(\Gamma)$ are in bijective linear correspondence to* 1*-pluricanonical forms on $X(\Gamma)$, that is, sections of the canonical bundle $\Omega_{X(\Gamma)}$. In particular, we have an isomorphism*

$$\Omega_{X(\Gamma)} \cong \mathcal{L}^{\otimes n}$$

and the vector space of orthogonal modular forms of weight nk can be identified with the space of global sections of $\Omega_X^{\otimes k}$ of X.

This is a first hint for the importance and naturality of orthogonal modular forms in the theory of orthogonal Shimura varieties: One can use the existence of a large supply of certain modular forms for Γ to determine the *Kodaira dimension* of $X(\Gamma)$ which is a basic algebraic invariant, roughly measuring the complexity of the variety.
It is worthwhile to consider the following classic special case of orthogonal modular forms which highlights the relation to the preceding chapter:

Elliptic modular forms

Choosing the lattice L as $\mathbb{Z}^3$ with quadratic form

$$q(x_0, x_1, x_2) = -x_0x_2 + x_1^2$$

of signature $(2, 1)$, the concept of orthogonal modular forms reduces to the well-known theory of elliptic modular forms: The tube domain model gives the symmetric space $\mathcal{D}_L$ as one of the connected components of

$$\mathbb{H}_L = \{Z_K \in K \otimes \mathbb{C} : q_K(Z_k) = Z_K^2 > 0\}$$

with $K \cong \mathbb{Z}$ one-dimensional and $q(k) = k^2$, so

$$\mathcal{D}_L \cong \mathbb{H}$$

can be thought of as the complex upper half-plane. The action of

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \langle \mathrm{O}(L), -\mathrm{id}\rangle \cong \mathrm{SL}_2(\mathbb{Z})$$

on a $\tau \in \mathbb{H}$ is given by the well-known Möbius transformation

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).\tau = \frac{a\tau + b}{c\tau + d}$$

and one can easily compute: A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a meromorphic modular form of weight k and multiplier system (character) χ (with a certain growth behavior) if and only if

$$f\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).\tau\right) = \chi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)(c\tau + d)^{2k} f(\tau).$$

Such modular forms are sometimes called *elliptic* (due to their role in the theory of elliptic curves) or classical. The space of weight k meromorphic modular forms is denoted by $M_k(\mathrm{SL}_2(\mathbb{Z}))$, the space of cusp forms (those vanishing for $\operatorname{Im}\tau \to \infty$)) by $S_k(\mathrm{SL}_2(\mathbb{Z}))$.
The condition on the growth behavior of f is due to the failure of the Koecher principle for $n = 1$; we will not go further into this.
In particular $f(\tau + 1) = f(\tau)$ and $f(-1/\tau) = \tau^{2k} f(\tau)$ for all $\tau \in \mathbb{H}$ as $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ generate $\mathrm{SL}_2(\mathbb{Z})$. Furthermore, the direct sum $\bigoplus_k \mathcal{M}_k$ is a graded $\mathbb{C}$-algebra by pointwise multiplication.
This is a non-empty theory, as the following number-theoretical examples show: The most classical elliptic modular forms are the *Eisenstein series*.

Definition 7.1.4. Let $k \geq 2$ be an integer and define

$$G_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m\tau+n)^{2k}}.$$

This converges absolutely on the upper half-plane and is an elliptic modular form of weight $2k$ in the sense above. The normalized Eisenstein series are

$$E_{2k} = \frac{1}{2\zeta(2k)} G_{2k} = (-1)^{k+1} \frac{(2k)!}{(2\pi)^{2k} B_{2k}} G_{2k}$$

with the Bernoulli numbers B_{2k} and have the Fourier expansion

$$E_{2k} = 1 + \frac{-4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

with $q = e^{2\pi i \tau}$. For $k = 2, 3$ this yields

$$E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + \mathcal{O}\left(q^4\right)$$

and

$$E_6(\tau) = 1 - 504q - 16632q^2 - 122976q^3 + \mathcal{O}\left(q^4\right).$$

Another modular form is the modular invariant

$$\Delta(\tau) = \frac{E_4^3(\tau) - E_6^2(\tau)}{1728} = q - 24q^2 + 72q^3 + \mathcal{O}\left(q^4\right) = \sum_{n\geq 1} \tau(n) q^n$$

of weight 12, $\tau(n)$ denoting the Ramanujan's tau function. By construction it vanishes at $i\infty$, so it is a cusp form.

For $k = 1$ the series in the definition of Eisenstein series does not converge well enough. However, it can be defined as the analytic continuation of

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{\operatorname{Im}(\tau)^s}{(m\tau+n)^2 |m\tau+n|^s}$$

for $s \to 0$. The resulting function $E_2 : \mathbb{H} \to \mathbb{C}$ is still holomorphic and satisfies

$$E_2(\tau+1) = E_2(\tau)$$

but its behavior under the transformation $\tau \mapsto -1/\tau$ is

$$E_1(-1/\tau) = \tau^2 E_2(\tau) - \frac{6i}{\pi}\tau.$$

By translation invariance it still has a Fourier expansion

$$E_2(\tau) = 1 - 24 \sum_{n\geq 1} \sigma_1(n) q^n = 1 - 24q - 72q^2 + \mathcal{O}\left(q^3\right).$$

Even if this is not a classical modular form due to its more general transformation behavior, it has tremendous importance for the ensuing theory of orthogonal modular forms.
The structure of the ring of holomorphic elliptic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ is well-understood: It is isomorphic (as a $\mathbb{C}$-algebra) to $\mathbb{C}[E_4, E_6]$. The space S_k is empty unless $k \geq 12$ and one has

$$S_k = \Delta \cdot M_{k-12}.$$

Moreover, one can consider more general modular forms by requiring modular transformation behavior only under Möbius transformations induced by elements of certain subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. For a good treatment of this well-developed theory see [DS06].
We transfer the characterization of modular forms as sections of line bundles from proposition 7.1.3 to this setting:

Lemma 7.1.5. *Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be an arithmetic subgroup acting freely on $\mathbb{H}$ and consider $X = \Gamma\backslash\mathbb{H}$. Let Ω^1_X be the canonical bundle on X.*
There is a bijective linear correspondence of elliptic meromorphic modular forms of weight $2k$ and global sections of $(\Omega^n_X)^{\otimes k}$.

Due to this we can consider $(\Omega^n_X)^{\otimes k}$ as the bundle defined by modular forms of weight $2k$, or, by abuse of notation, as *the bundle of modular forms of weight* $2k$ and denote it by the slightly misleading notation $M_{2k}(\Gamma)$. It should be clear from context whether this means the line bundle or its space of sections.

Vector-valued modular forms

It is a lot harder too see that the theory of general orthogonal modular forms is as rich as it is, but due to the seminal work of Borcherds, there is a very powerful machinery for the construction of orthogonal modular forms: the *Borcherds lift* which lifts certain further generalizations of elliptic modular forms to orthogonal modular forms which are usually called *Borcherds products*.
We give a short review of the necessary terminology to introduce this lift, starting with the input function, the so called *vector-valued modular forms*. We will largely follow the treatment in [Bru02].
Let D be a discriminant form. We define $\mathbb{C}[D]$ as the $\mathbb{C}$-vector space of all formal $\mathbb{C}$-linear combinations of basis elements $\mathbf{e}_\gamma$ for $\gamma \in D$. As known, the modular group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and we can define the following representation of this group on $\mathbb{C}[D]$ via suitable definition of the action of these two matrices.

Definition 7.1.6. The definitions

$$\rho_D(T)\mathbf{e}_\gamma = e(Q(\gamma))\mathbf{e}_\gamma$$

and

$$\rho_D(S)\mathbf{e}_\gamma = \frac{e(-\operatorname{sgn}(D)/8)}{\sqrt{|D|}} \sum_{\delta\in D} e(-(\gamma,\delta))\mathbf{e}_\delta$$

for every $\gamma \in D$ and its continuation to $\mathrm{SL}_2(\mathbb{Z})$ define a representation ρ_D of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$, the *Weil representation.*

A good intuition about this is to think of the action of S as a kind of Fourier transformation on $\mathbb{C}[D]$.
It is not at all clear that these definitions indeed extend to the group. It may be checked by a standard computation using the known presentation of $\mathrm{SL}_2(\mathbb{Z})$.
While the above definition appears to be rather ad-hoc, this is a special case of the very natural general Weil representation: It can be obtained by pulling back the general Weil representation of the metaplectic group to $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{O}(2,n)$.
N.B.: The factors of this product form a dual reductive pair; this is the deeper reason for the Borcherds lift in the next section to work. The interested reader may consult [Kud94].

Definition 7.1.7. Let D be a discriminant form of level $N \in \mathbb{N}$, even signature and ρ_D its Weil representation. Let $F : \mathbb{H} \to \mathbb{C}[D]$ be a function and $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. We define

$$F|_\alpha(\tau) := \det(\alpha)^{k/2} j(\alpha,t)^{-k} \rho_D(\alpha)^{-1} F(\alpha.\tau).$$

With these definitions we introduce the notion of *vector-valued modular forms*:

Definition 7.1.8. Let D be a discriminant form of level $N \in \mathbb{N}$, even signature and ρ_D its Weil representation. Let $F : \mathbb{H} \to \mathbb{C}[D]$ be a function.
Any such function can be written as $F = \sum_{\gamma\in D} F_\gamma \mathbf{e}_\gamma$ for $F_\gamma : \mathbb{H} \to \mathbb{C}$. It is called a *vector-valued modular form of weight k and representation ρ_D*, if it has the following properties:

i) F is holomorphic on $\mathbb{H}$.

ii) F is modular w.r.t ρ_D, i.e. $F|_\alpha(\tau) = F(\tau)$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

iii) F is holomorphic at the cusps for every $\gamma \in D$, i.e. for $\operatorname{Im}\tau \to \infty$ and every $M \in \mathrm{SL}_2(\mathbb{Z})$ the absolute value of $F|_M(\tau)$ stays bounded.

We denote the vector space of vector-valued modular forms of weight k by $\mathcal{M}_k(\rho_D)$. If $F_\gamma(\tau) \to 0$ for $\operatorname{Im}\tau \to \infty$ and every $\gamma \in D$, we call F a cusp form of weight k; the vector space of cusp forms of given weight k is denoted by $\mathcal{S}_k(\rho_D)$.

A slight generalization of this will be of even greater importance:

Definition 7.1.9. A function $F : \mathbb{H} \to \mathbb{C}[D]$ as above with iii) substituted by the weaker condition

iii′) F has at most a pole at ∞

is called a *nearly holomorphic* modular form. The space of these forms for given weight k is denoted by $\mathcal{M}^!_k(\rho_D)$.

We have the inclusions

$$\mathcal{M}^!_k(\rho_D) \subseteq \mathcal{M}_k(\rho_D) \subseteq \mathcal{S}_k(\rho_D).$$

Any nearly holomorphic vector-valued modular form

$$F = \sum_{\gamma \in D} F_\gamma \mathbf{e}_\gamma \in \mathcal{M}_k(\rho_D)$$

has a Fourier expansion of the form

$$F = \sum_{\gamma \in D} F_\gamma \mathbf{e}_\gamma = \sum_{\gamma \in D} \sum_{n=n_0(\gamma)}^{\infty} c_\gamma(n) q^{n/N} \mathbf{e}_\gamma$$

where $n \in q(\gamma) + \mathbb{Z}$. We call $c_\gamma(n)$ the n-th Fourier coefficient of F with respect to $\gamma \in D$. Condition (iii) is again equivalent to $n_0(\gamma) \geq 0$ for all $\gamma \in D$ and F is a cusp form if and only if $c_\gamma(n) = 0$ for all $\gamma \in D$ and $n \leq 0$.
The finite polynomial

$$\sum_{\gamma \in D} F_\gamma \mathbf{e}_\gamma = \sum_{\gamma \in D} \sum_{n<0} c_\gamma(n) q^{n/N} \mathbf{e}_\gamma$$

is called the *principal part of F*.
If the discriminant group is trivial, this is exactly the same as an ordinary elliptic modular form.
A natural class of true vector-valued modular forms are given by the theta series of even definite lattices:

Example 7.1.10. Let L be a positive definite even lattice with discriminant form Δ_L and quadratic form q_L. For $\gamma = v + L \in \Delta_L$ define the theta series

$$\theta_\gamma(\tau) = \sum_{l \in \gamma} q^{q_L(l)} = \sum_{n \in q(\gamma)+\mathbb{Z}} c_\gamma(n) q^n$$

with $c_\gamma(n)$ the number of vectors in γ of norm $2n$ (finite by definiteness). The sum

$$\Theta_L(\tau) = \sum_{\gamma \in \Delta_L} \theta_\gamma(\tau) \mathbf{e}_\gamma$$

is a vector-valued modular form for the Weil representation ρ_{Δ_L} of Δ_L.

For unimodular lattices L, these vector-valued modular forms are again elliptic of weight $\mathrm{rank}(L)/2$ and may coincide with Eisenstein series: For $L = \mathrm{E}_8$ the theta series Θ_{E_8} is just the Eisenstein series E_4; similarly $\Theta_\Lambda = E_{12}$ for Λ the Leech lattice.
There are several mechanisms to lift vector-valued modular forms between different discriminant forms. We know from proposition 5.1.12 that any inclusion $M \subset M'$ of even lattices corresponds to an isotropic subgroup $I \subset \Delta_M$ with $I^\perp/I \cong \Delta_{M'}$ and every overlattice of M arises in this way. We have the canonical map $\pi : I^\perp \to \Delta_{M'}$ and can use this to define maps between the group algebras $\mathbb{C}[\Delta_M]$ and $\mathbb{C}[\Delta_{M'}]$ as follows:

Proposition 7.1.11. *Let $M \subset M'$ be even lattices. Define maps*

$$\uparrow^M_{M'} \colon \mathbb{C}[\Delta_{M'}] \to \mathbb{C}[\Delta_M]$$

by the linear continuation of $\boldsymbol{e}_\gamma \mapsto \sum_{\delta \in \pi^{-1}(\gamma)} \boldsymbol{e}_\delta$ and

$$\downarrow^M_{M'} \colon \mathbb{C}[\Delta_M] \to \mathbb{C}[\Delta_{M'}]$$

by the linear continuation of $\boldsymbol{e}_\gamma \mapsto \begin{cases} \boldsymbol{e}_{\pi(\gamma)} & \gamma \in I^\perp \\ 0 & \gamma \notin I^\perp \end{cases}.$
These linear operators define linear maps

$$\uparrow^M_{M'} \colon M^!_k(\rho_{\Delta_{M'}}) \to M^!_k(\rho_{\Delta_M})$$

and

$$\downarrow^M_{M'} \colon M^!_k(\rho_{\Delta_M}) \to M^!_k(\rho_{\Delta_{M'}})$$

on the spaces of nearly holomorphic vector-valued modular forms of given weight k.

We give the simplest example of this situation:

Example 7.1.12. Let L be an unimodular lattice and $K \subseteq L$ any even sublattice, then $\Delta_L = \{0\}$ and $\uparrow^K_L \colon M_k(\mathrm{SL}_2(\mathbb{Z})) \to M_k(\rho_{\Delta_K})$ maps meromorphic modular forms of weight k to vector-valued modular forms for ρ_{Δ_K}. If the Fourier series of $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ is given by

$$f(\tau) = \sum_n c_n q^n$$

any component function F_γ of $F = \uparrow^K_L (f)$ has this Fourier expansion.

There is another operation on vector-valued modular forms, cf. [Ma19, Section 3]: Let $L = M \oplus K$ with K being definite, then the corresponding discriminant forms satisfy $\Delta_L \cong \Delta_M \oplus \Delta_K$. Consider a nearly holomorphic vector-valued modular form

$$F = \sum_{\gamma \in \Delta_L} F_\gamma \mathbf{e}_\gamma = \sum_{\gamma_M \in \Delta_M} \sum_{\gamma_K \in \Delta_K} F_{(\gamma_M, \gamma_K)} \mathbf{e}_{(\gamma_M, \gamma_K)}$$

of weight k with respect to ρ_{Δ_L}. Let $\Theta_K = \sum_{\gamma_k \in \Delta_K} \theta_{\gamma_K} \mathbf{e}_{\gamma_K}$ be the theta series of K as in example 7.1.10. The Θ-*contraction* of F is given by

$$\langle F, \Theta_K \rangle = \sum_{\gamma_M \in \Delta_M} \left(\sum_{\gamma_K \in \Delta_K} F_{(\gamma_M, \gamma_K)} \theta_{\gamma_K} \right) \mathbf{e}_{\gamma_M}$$

and is a nearly holomorphic modular form of weight $k + \mathrm{rank}(K)/2$ with respect to ρ_{Δ_M}. This operation preserves the integrality of the principal part and, under certain circumstances, the constant term of $\langle F, \Theta_K \rangle$ remains even. For details and proofs see [Ma19, Lemma 3.3].

7.2. Borcherds products

In [Bor98] Borcherds gives a mechanism to turn suitable vector-valued modular forms for the Weil representation of L into orthogonal modular forms on $\Gamma\backslash\mathcal{D}_L$. Moreover, these orthogonal modular forms have a very special product expansion and divisor.
To be able to describe the divisor of the Borcherds products, we introduce the notion of *Heegner divisors*:
Consider the projective model $\mathcal{K}^+$ of the symmetric space as in section 6.1 for a fixed even lattice L of signature $(2, n)$.

Definition 7.2.1. Let $\lambda \in L'$ be a primitive vector of negative norm $q(\lambda) = m < 0$ with class $\lambda' \in [\beta] = \beta + L$ in L'/L. The set

$$\lambda^\perp = \{[Z] \in \mathcal{K}^+ |\, [Z, \lambda] = 0\}$$

is a prime divisor on $\mathcal{K}^+$. The divisor

$$\sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \lambda^\perp$$

is called the *Heegner divisor* $H(\beta, m)$ on $\mathcal{D}$.

This divisor is invariant under any $\Gamma \subseteq \widetilde{\mathrm{SO}}^+(L)$, so we will denote its image in the locally symmetric space $\Gamma\backslash\mathcal{D}$ by the same notation.
The famous theorem of Borcherds on the existence and construction of orthogonal modular forms can be stated as follows, cf. [Bor98, Theorem 13.3]:

Theorem 7.2.2. *Let L be an even lattice of signature $(2, n)$, $n > 2$ even with discriminant form Δ_L and $\Gamma \subseteq \widetilde{\mathrm{SO}}^+(L)$ a finite index subgroup. Let F be a nearly holomorphic modular form for the Weil representation ρ_D of Γ on $\mathbb{C}[\Delta_L]$ with weight $1 - n/2$. Assume that $c_0(0)$ is even and $c_\gamma(m) \in \mathbb{Z}$ for $m < 0$. Then there exists an orthogonal automorphic form $\Psi_F : \Gamma\backslash\mathcal{D}_L \to \mathbb{C}$ of weight $c_0(0)/2$ with divisor*

$$\frac{1}{2} \sum_{\mu\in\Delta_L} \sum_{0>m\in\mu^2/2+\mathbb{Z}} c_\mu(m) H(\mu, m).$$

As a function on $\mathcal{D}_L$ the order of vanishing at a primitive $\lambda \in L$ is given by

$$\sum_{\substack{x\in\mathbb{Q}\\ x\lambda\in L'}} c_{xy+L}(q(x\lambda)).$$

In the tube domain model, there is a very concrete description of Ψ_F as a product which is responsible for the name.

Example 7.2.3. We give two well-studied examples in this theory:

i) Let $L = II_{2,26}$ be the unique even unimodular lattice of dimension 28, i.e.

$$L \cong \Lambda(-1) \oplus II_{1,1} \oplus II_{1,1} = K \oplus II_{1,1}$$

for Λ the Leech lattice and $K = II_{1,25}$. We have $L' = L$, so the Weil representation is trivial and an input of the Borcherds lifting has to be a meromorphic elliptic modular form of weight $1 - 26/2 = -12$, so we can consider the lifting of the elliptic form

$$\frac{1}{\Delta} = q^{-1} + 24 + 324q + \mathcal{O}\left(q^2\right) = \sum_{n \geq 0} c_n q^{n-1}$$

of weight -12 with principal part q^{-1}. Its lift $\Psi =: \Phi_{12}$ is an orthogonal modular form of weight 12 and divisor $H(-1)$ for the group $\Gamma = \mathrm{SO}^+(II_{2,26})$. In the tube domain realization corresponding to the decomposition $II_{2,26} = K \oplus II_{1,1}$ as above, the function is given as

$$\Psi(Z) = e((\varrho, Z)) \prod_{\lambda \in K^+} (1 - e((\lambda, Z)))^{c_{1-q(\lambda)}},$$

with $e(z) = e^{2\pi i z}$, K^+ denoting a certain sublattice of K and ϱ the *Weyl vector*.

ii) Let $L = II_{1,9}$ be the unique even unimodular lattice of dimension 12. The input of the Borcherds lift has to be a meromorphic modular form of weight $1 - 10/2 = -4$ for the trivial Weil representation and we can consider the lifting of

$$\frac{E_4^2}{\Delta} = q^{-1} + 504 + 16405q + \mathcal{O}(q^2)$$

which is an orthogonal modular form of weight 252, again with divisor $H(-1)$ (but of course on a completely different symmetric space).

The especially simple form of their divisors make these orthogonal modular form *reflective*, a property that will be of tremendous importance later on in section 11.3.

Borcherds products for commensurable groups

The Borcherds lift to a lattice L only produces orthogonal modular forms with respect to the discriminant kernel $\widetilde{\mathrm{SO}}^+(L) \subseteq \mathrm{O}(L)$. These are also modular forms with respect to any subgroup $\Gamma \subseteq \widetilde{\mathrm{SO}}^+(L)$; moreover, they can be used to define modular forms for any finite index-subgroup $\Gamma' \subseteq \mathrm{SO}^+(L)$ as follows:
Let Ψ be an orthogonal modular form on $\mathcal{D}_L$ with respect to Γ. Choose representatives $\gamma_1, \ldots, \gamma_l$ of $\Gamma \cap \Gamma' \backslash \Gamma$ and consider the symmetrization

$$\Psi' = \prod_{i=1}^{l} (\Psi \circ \gamma_i).$$

This is a modular form for Γ'.

Thinking back to the different models for the symmetric space this gives a new perspective: If $K \subseteq L$ are even lattices of the same dimension $n+2$, an orthogonal modular form Ψ_L of weight k on $\mathcal{D}_L$ with respect to $\Gamma(L) = \widetilde{\mathrm{SO}}^+(L)$ corresponds to a section of the canonical bundle ${\Omega^n_{X(L)}}^{\otimes k}$ and defines an orthogonal modular form Ψ_K on $\mathcal{D}_K$ with respect to $\Gamma(K) = \widetilde{\mathrm{SO}}^+(K)$, which is itself a section of ${\Omega^n_{X(K)}}^{\otimes k}$. Considered as functions on the generalized upper half-space $\mathbb{H}_K \cong \mathbb{H}_L$ or the corresponding projective model, these modular forms obviously coincide.
This may be indicative that it is more natural for us to treat these forms rather as 'well-defined objects' on the locally symmetric space $\Gamma(L)\backslash\mathcal{D}_L$ than as analytic functions on $\mathcal{D}_L$ with certain symmetries, as this exhibits the difference between Ψ_L and Ψ_K more clearly. We will switch back and forth between these two views whenever necessary.

Pulling back Borcherds products

It is possible to restrict Borcherds products to irreducible components of Heegner divisor as shown in this section. We closely follow the treatment in [Ma19].
Let $K \subseteq L$ a primitive, negative definite sublattice and denote by M its orthogonal complement

$$M = K^\perp \cap L$$

inside L. Assume further that the Witt index of M is smaller than $\mathrm{rank}(M) - 2$ (to ensure the applicability of the Koecher principle which is an central ingredient of the proof).
Let F be a nearly holomorphic modular form of weight $1 - {n}/{2}$ as in theorem 7.2.2 with the corresponding Borcherds product $\Psi = \Psi_F$ considered as a modular function $\mathcal{D}_L \to \mathbb{C}$. We define a function

$$\Psi\|_{\mathcal{D}_M} : \mathcal{D}_M \to \mathbb{C}$$

as follows:

Definition 7.2.4. Let $\lambda \in K$ be a primitive vector and denote the order of Ψ on the divisor $\lambda^\perp$ (cf. definition 7.2.1) by $d(\lambda)$. Let $(\cdot, \lambda) : \mathcal{D}_L \to \mathbb{C}, v \mapsto (v, \lambda)$ be the functional induced by λ. Define

$$\Psi\|_{\mathcal{D}_M} = \left.\frac{\Psi}{\prod (\cdot, \lambda)^{d(\lambda)}}\right|_{\mathcal{D}_M}$$

with $|_{\mathcal{D}_M}$ denoting the actual restriction and the product running over primitive $\lambda \in K$, choosing one of $\pm\lambda$.

The product in the denominator is finite as the definite lattice $(K \otimes \mathbb{Q}) \cap L'$ has only finitely many primitive vectors of a given norm and there are only finitely many non-vanishing Fourier coefficients in the principal part of F.
Note that this function is indeed a *quasi-pullback* instead of a simple pullback: As the domain of $\Psi\|_{\mathcal{D}_M}$ may be contained in the divisor, a true pullback may result in the zero function; the exponents d in the quasi-pullback are chosen such that the resulting quotient is non-zero and well-defined on $\mathcal{D}_M$.

The quasi-pullback is again an orthogonal modular form, this time corresponding to the lattice M and its discriminant kernel $\widetilde{\mathrm{SO}}^+(M)$:

Proposition 7.2.5 ([Bor98])**.** *The quasi-pullback $\Psi\|_{\mathcal{D}_M}$ is a non-zero meromorphic modular form on $\mathcal{D}_M$ with respect to $\widetilde{\mathrm{SO}}^+(M)$ and has weight $\mathrm{wt}(\Psi) + \sum d(\lambda)$ where $\mathrm{wt}(\Psi)$ is the weight of Ψ and the sum runs over primitive $\lambda \in K$, choosing exactly one of $\pm\lambda$.*

Remembering proposition 7.1.3, this can also be considered as a pluricanonical form for $\widetilde{\mathrm{SO}}^+(M)$.
By the theory developed by Ma in [Ma19], this quasi-pullback $\Psi\|_{\mathcal{D}_M}$ itself can be obtained as the Borcherds lift of a nearly holomorphic modular form F' which can be constructed in terms of F and the lattice K.

Theorem 7.2.6 ([Ma19, Theorem 1.1])**.** *Up to a constant, $\Psi\|_{\mathcal{D}_M}$ is the Borcherds lift $\Psi_{F'}$ of the nearly holomorphic modular form*

$$F' = \langle \uparrow_L^{M\oplus K} F, \Theta_K \rangle.$$

We will use this theory later on to construct a plethora of Borcherds products on embedded Shimura varieties of orthogonal type.
As the principal part of $F' = \langle \uparrow_L^{M\oplus K} F, \Theta_K \rangle$ can be computed easily from the principal part of F, one can compute the divisor of F' as well: By the definition of Heegner divisors we can write the divisor

$$\frac{1}{2} \sum_{\mu \in \Delta_L} \sum_{0 < m \in \mu^2/2 + \mathbb{Z}} c_\mu(m) H(\mu, m)$$

on the symmetric domain $\mathcal{D}_L$ equivalently as

$$\sum_{\substack{l \in L'/\pm 1 \\ q(l) < 0}} c_{l+L}(q(l)) \left(l^\perp \cap \mathcal{D}_L \right).$$

By [Ma19, Proposition 3.4 and Lemma 3.6] the divisor of $\Psi(F')$ on $\mathcal{D}_M$ is given (regardless of being in the split or general case) by

$$\sum_{\substack{m \in M'/\pm 1 \\ q(m) < 0}} c'_{m+M}(q(m)) \left(m^\perp \cap \mathcal{D}_M \right)$$

with $c'_{m+M}(q(m))$ denoting the Fourier coefficients of principal part of the lift $\uparrow_L^{M\oplus K} (F)$.
Note that, in general, the quasi-pullback is a pullback to an irreducible component of a Heegner divisor, not to the Heegner divisor itself.
This end our exposition of orthogonal modular forms. In the next chapter we will return to the geometric point of view and apply the general theory of toroidal compactifications to the case of orthogonal locally symmetric spaces.

8. Toroidal compactifications in the orthogonal case

In this chapter we will translate the construction process for toroidal compactifications to the setting of orthogonal locally symmetric spaces $X(\Gamma) = \Gamma\backslash\mathcal{D}_L$ with L an even lattice of signature $(2, n)$ of level N and $\Gamma \subseteq \mathrm{O}(L)$ arithmetic. We assume further that any primitive isotropic vector $l \in L$ satisfies $(l, L) = N\mathbb{Z}$.
We recall from chapter 3 that the input data for a toroidal compactification of $\Gamma\backslash\mathcal{D}_L$ is a Γ-admissible family, that is, a family $\Sigma = \{\Sigma(\mathcal{F})|\mathcal{F}\}$ of cone decompositions indexed by the rational boundary components $\mathcal{F}$ of $\mathcal{D}_L$ such that $\Sigma(\mathcal{F})$ is a $\overline{\mathcal{P}(\mathcal{F})_{\mathbb{Z}}}$-admissible decomposition of $\overline{\mathcal{C}(\mathcal{F})}^{\mathrm{rat}}$ and the collection Σ satisfies the compatibility conditions

i) If $\mathcal{F}_2 = \gamma\mathcal{F}_1$ for some $\gamma \in \Gamma$, then $\gamma\Sigma(\mathcal{F}_1) = \Sigma(\mathcal{F}_2)$

ii) If $\mathcal{F}_1 \subseteq \overline{\mathcal{F}_2}$, then $\Sigma(\mathcal{F}_1) = \left\{\sigma \cap \overline{\mathcal{C}(\mathcal{F}_1)}^{\mathrm{rat}} \,|\, \sigma \in \Sigma(\mathcal{F}_2)\right\}$.

We will see that these conditions can be made more explicit and simple if $\mathcal{D}_L$ is an orthogonal symmetric space.
The first section of this chapter will treat the structure of the integral parabolic group $\mathcal{P}(\mathcal{F})_{\mathbb{Z}}$ and its related objects (e.g. $\overline{\mathcal{P}(\mathcal{F})_{\mathbb{Z}}}$); in the second section we will make the necessary combinatorial data for the construction of toroidal compactifications in the orthogonal case explicit and describe a special symmetric choice of this input. We conclude with a short characterization of the types of singularities of toroidal compactifications of non-smooth orthogonal locally symmetric spaces.

8.1. Integral parabolics

We described the structure of the maximal real parabolics corresponding to rational boundary components in section 6.3. The structure of the intersection of these parabolics with Γ is of importance, so we describe the Langlands decompositions of these *integral parabolics* for $\Gamma = \widetilde{\mathrm{SO}}^+(L)$. For the structure theory in the general (i.e. non-orthogonal) case, see [Zem20].
We recall that rational boundary components of $\mathcal{D}_L$ are one- or zero-dimensional; corresponding to any choice $\mathcal{F}$ there is a choice of coordinates for L as in section 6.3.
Let $\mathcal{F}_0$ be a zero-dimensional rational boundary component of $\mathcal{D}_L$ with corresponding choice of basis. This induces a decomposition

$$L = II_{1,1}(N) \oplus K$$

for a lattice K of signature $(1, n-1)$.
The intersections of $\widetilde{\mathrm{SO}}^+(L)$ with the objects related to $\mathcal{P}(\mathcal{F})$ are as follows:

$$\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}} \cong K \rtimes \widetilde{\mathrm{SO}}^+(K)$$

with trivial $G_h(\mathcal{F}_0)_{\mathbb{Z}} \cong \mathcal{V}(\mathcal{F}_0)_{\mathbb{Z}} \cong \mathcal{M}(\mathcal{F}_0)_{\mathbb{Z}}$, the lattice

$$\mathcal{U}(\mathcal{F}_0)_{\mathbb{Z}} \cong K$$

and

$$G_l(\mathcal{F}_0)_{\mathbb{Z}} \cong \widetilde{\mathrm{SO}}^+(K) \cong \mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}/\mathcal{U}(\mathcal{F}_0)_{\mathbb{Z}}.$$

The image of the last group $\overline{\mathcal{P}(\mathcal{F}_0)}_{\mathbb{Z}} \cong G_l(\mathcal{F}_0)_{\mathbb{Z}}$ in $\mathrm{Aut}(\mathcal{C}(\mathcal{F}_0))$ is an arithmetic subgroup.
An analogous treatment of the case of a one-dimensional rational boundary component $\mathcal{F}_1$ is more complicated and can be seen in [Zem20, Proposition 3.3]. There is a fundamental short exact sequence:

$$1 \to \tilde{H}(\Lambda, \mathbb{Z}) \to \mathcal{P}(\mathcal{F}_1)_{\mathbb{Z}} \to \Gamma(N) \to 1$$

with $\tilde{H}(\Lambda, \mathbb{Z})$ a certain subgroup of the integral Heisenberg group and $\Gamma(N)$ the N-th principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.
We give only a description of $\overline{\mathcal{P}(\mathcal{F}_1)}_{\mathbb{Z}}$ as this group is most important for our considerations.
Let $\mathcal{F}_1$ be a one-dimensional rational boundary component of $\mathcal{D}_L$ with corresponding choice of basis. This induces a decomposition

$$L = II_{1,1}(N) \oplus II_{1,1}(N) \oplus \Lambda$$

for a negative definite lattice Λ of signature $(0, n-2)$. Then:

$$\overline{\mathcal{P}(\mathcal{F}_1)}_{\mathbb{Z}} \cong (\Lambda \times \Lambda) \rtimes \Gamma(N).$$

This can be shown by direct calculation or by the general methods in [Zem20].
Let $\mathcal{F}_0 \subset \overline{\mathcal{F}_1}$ be an inclusion of adjoint rational boundary components. In this case, the basis of L can be chosen such that

$$L = II_{1,1}(N) \oplus K \text{ and } K = II_{1,1}(N) \oplus \Lambda.$$

Later on, an important role will be played by the intersection $\mathcal{P}(\mathcal{F}_0) \cap \mathcal{P}(\mathcal{F}_1)$ and its quotients by $\mathcal{U}(\mathcal{F}_0)$ and $\mathcal{U}(\mathcal{F}_1)$ resp. their integral counterparts.
A simple calculation shows that

$$\begin{aligned}
\mathcal{P}(\mathcal{F}_0) \cap \mathcal{P}(\mathcal{F}_1)/\mathcal{U}(\mathcal{F}_1) &\cong \mathbb{R} \rtimes \left(\mathbb{R}^{n-2} \times \mathbb{R}^{n-2}\right) \\
&\subseteq \mathrm{SL}_2(\mathbb{R}) \rtimes \left(\mathbb{R}^{n-2} \times \mathbb{R}^{n-2}\right) \\
&\cong \mathcal{P}(\mathcal{F}_1)/\mathcal{U}(\mathcal{F}_1)
\end{aligned}$$

with $\mathbb{R} \subseteq \mathrm{SL}_2(\mathbb{R})$ via the monomorphism $r \mapsto \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)$. The inclusion into $\mathcal{P}(\mathcal{F}_0)/\mathcal{U}(\mathcal{F}_0)$ is a bit more complicated and not needed here. The integral counterpart is

$$\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}} \cap \mathcal{P}(\mathcal{F}_1)_{\mathbb{Z}}/\mathcal{U}(\mathcal{F}_1)_{\mathbb{Z}} \cong \mathbb{Z} \rtimes (\Lambda \times \Lambda)$$

with the $\mathbb{Z}$-factor realized as the subgroup of $\Gamma(N)$ generated by $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$.
Analogously we find

$$\mathcal{P}(\mathcal{F}_0) \cap \mathcal{P}(\mathcal{F}_1)/\mathcal{U}(\mathcal{F}_0) \cong \mathbb{R}^{n-2}$$

and

$$\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}} \cap \mathcal{P}(\mathcal{F}_1)_{\mathbb{Z}}/\mathcal{U}(\mathcal{F}_0)_{\mathbb{Z}} \cong \Lambda.$$

Remark 8.1.1. It is no coincidence that the groups related to one-dimensional boundary components look remarkably similar to those appearing in the context of the toroidal compactification of the universal elliptic curve and its fiber products:
The boundary components of the toroidal compactifications of orthogonal locally symmetric spaces added with respect to one-dimensional boundary components will turn out to be exactly these Kuga-Sato varieties.

Once again we want to clarify what happens with these objects under the identification $\mathcal{D}_L \cong \mathcal{D}_{L(M)}$ for $M \geq 1$ (unfamiliar notation, but necessary to distinguish it from the assumed level N of L in this section):
We have seen before that the real structures are compatible with this identification, but the rational structures differ by a factor of $\sqrt{M}$, e.g.

$$\mathcal{U}(\mathcal{F}'_0) = \mathcal{U}(\mathcal{F}_0)$$

and

$$\mathcal{U}(\mathcal{F}'_0)_{\mathbb{Z}} = K(M) \cong K = \mathcal{U}(\mathcal{F}_0)_{\mathbb{Z}}.$$

Here $\mathcal{F}'_0$ is the rational zero-dimensional boundary component of $\mathcal{D}_{L(M)}$ corresponding to the zero-dimensional rational boundary component $\mathcal{F}_0$ of $\mathcal{D}_L$ under this identification; the image of the embedding is

$$\mathcal{U}(\mathcal{F}'_0)_{\mathbb{Z}} \cong K(M) \hookrightarrow \sqrt{M}K \subseteq K \otimes \mathbb{R} \cong \mathcal{U}(\mathcal{F}_0);$$

analogously for the other objects. We should also mention that the rational closures $\overline{\mathcal{C}(\mathcal{F})}^{\mathrm{rat}}$ and $\overline{\mathcal{C}(\mathcal{F}')}^{\mathrm{rat}}$ coincide under this identification since a rational boundary component of $\overline{\mathcal{C}(\mathcal{F})}^{\mathrm{rat}}$ corresponds to a primitive isotropic vector in $f \in K$. These correspond under the bijection $f \mapsto \sqrt{M}f \in K(M)$ to primitive isotropic vectors in $K(M) \cong \sqrt{M}K \subseteq K \otimes \mathbb{R}$ which gives a rational boundary component of $\overline{\mathcal{C}(\mathcal{F}')}^{\mathrm{rat}}$, so the cones get identified via the identification induced from $\mathcal{D}_L \cong \mathcal{D}_{L(M)}$.
For a rational zero-dimensional boundary component $\mathcal{F}_0$ the group

$$G_l(\mathcal{F}'_0)_{\mathbb{Z}} = G_l(\mathcal{F}'_0) \cap \widetilde{\mathrm{SO}}^+(L(M)) \cong \widetilde{\mathrm{SO}}^+(K(M))$$

is the M-th principal congruence subgroup of

$$G_l(\mathcal{F}_0)_{\mathbb{Z}} = G_l(\mathcal{F}_0) \cap \widetilde{\mathrm{SO}}^+(L) \cong \widetilde{\mathrm{SO}}^+(K),$$

since

$$\mathrm{SO}^+(K(M)) \cong \mathrm{SO}^+\left(\sqrt{M}K\right) = \mathrm{SO}^+(K).$$

Here, the first isomorphism is via the underlying identification $\mathcal{D}_L \cong \mathcal{D}_{L(M)}$ and the second equality is an equality as automorphism groups of lattices in the common real vector space $K \otimes \mathbb{R}$ with the same bilinear form.

8.2. Toroidal data

With this overview of the relevant objects, we are ready to translate the general construction of toroidal compactifications to the orthogonal setting and assume $\Gamma \subseteq \mathrm{O}(L)$ (not necessarily equal to $\widetilde{\mathrm{SO}}^+(L)$) for the moment.
We recall again the notion of a general Γ-admissible family in definition 3.1.4 and note that condition ii) is trivially satisfied in the orthogonal case:
The condition $\mathcal{F}_1 \subseteq \overline{\mathcal{F}_2}$ implies either $\mathcal{F}_1 = \mathcal{F}_2$ and therefore trivially $\Sigma(\mathcal{F}_1) = \Sigma(\mathcal{F}_2)$ or $\mathcal{F}_1$ is zero-dimensional and $\mathcal{F}_2$ is one-dimensional, so $\overline{\mathcal{C}(\mathcal{F}_1)}^{\mathrm{rat}} \cong \mathbb{R}_{\geq 0}$ and

$$\Sigma(\mathcal{F}_2) \cap \overline{\mathcal{C}(\mathcal{F}_1)}^{\mathrm{rat}} = \{\{0\}, \mathbb{R}_{\geq 0}\} = \Sigma(\mathcal{F}_1)$$

in every case and the $\overline{\mathcal{P}(\mathcal{F}_i)_{\mathbb{Z}}}$-admissible cone decompositions $\Sigma(\mathcal{F}_1)$ and $\Sigma(\mathcal{F}_2)$ are compatible.
In particular: A toroidal compactification of $\mathcal{D}_L$ is determined by its $\overline{\mathcal{P}(\mathcal{F})_{\mathbb{Z}}}$-admissible cone decompositions of $\overline{\mathcal{C}(\mathcal{F})}^{\mathrm{rat}}$ for the zero-dimensional boundary components $\mathcal{F}$.
We can push this simplification even further:
Condition i) of definition 3.1.4 even shows that it suffices to fix as system $(\Sigma(\mathcal{F}))_{\mathcal{F}\in S}$ of $\overline{\mathcal{P}(\mathcal{F})_{\mathbb{Z}}}$-admissible cone decompositions for a system S of Γ-representatives of zero-dimensional boundary components $\mathcal{F}$ and take condition i) as a definition.

Construction 8.2.1. Let S be a system of Γ-representatives of zero-dimensional boundary components $\mathcal{F}$. For $\mathcal{F}_0 \in S$ and $\gamma \in \Gamma$ we define

$$\Sigma(\gamma\mathcal{F}_0) = \{\gamma\sigma | \sigma \in \Sigma(\mathcal{F}_0)\}.$$

An automorphism $\gamma \in \Gamma$ preserves polyhedrality and transforms $\overline{\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}}$-admissible cone decompositions of $\overline{\mathcal{C}(\mathcal{F}_0)}^{\mathrm{rat}}$ to $\overline{\mathcal{P}(\gamma\mathcal{F}_0)_{\mathbb{Z}}}$-admissible cone decompositions of $\overline{\mathcal{C}(\gamma\mathcal{F}_0)}^{\mathrm{rat}}$ (since it preserves trivially the decomposition, invariance and finiteness properties). Doing this for all $\mathcal{F} \in S$ gives a Γ-admissible family of cone decompositions by construction. We call it the Γ-*admissible family induced by* $(\Sigma(\mathcal{F}))_{\mathcal{F}\in S}$ and the corresponding toroidal compactification the *toroidal compactification induced by* $(\Sigma(\mathcal{F}))_{\mathcal{F}\in S}$.

If Γ is causing no singularities on $\Gamma\backslash\mathcal{D}_L$, smoothness of the inducing decompositions implies smoothness of the toroidal compactification:

Lemma 8.2.2. *Assume that Γ is neat and let Σ be a Γ-admissible family induced by a system $(\Sigma(\mathcal{F}))_{\mathcal{F}\in S}$ of admissible cone decompositions. Then: If for every $\mathcal{F} \in S$ and every $\sigma \in \Sigma_{\mathcal{F}}$ the cone σ is generated by a part of a basis of the lattice $\mathcal{U}(\mathcal{F})_{\mathbb{Z}}$, the toroidal compactification $\overline{X}_{\Sigma}^{tor}$ induced by $(\Sigma(\mathcal{F}))_{\mathcal{F}\in S}$ is smooth.*

Upon closer inspection we note that construction 8.2.1 can be applied even for systems of $\mathrm{O}^+(L)$-representatives of rational boundary component $\mathcal{F}$: A $\gamma \in \overline{\mathrm{O}^+(L)}$ still preserves polyhedrality and the decomposition property, as well as turning $\overline{\mathcal{P}(\mathcal{F})_{\mathbb{Z}}}$-invariance and -finiteness into $\overline{\mathcal{P}(\gamma\mathcal{F})_{\mathbb{Z}}}$-invariance and -finiteness, since $\gamma : \mathcal{U}(\mathcal{F}) \to \mathcal{U}(\gamma\mathcal{F})$ is an isomorphism of vector spaces mapping $\overline{\mathcal{C}(\mathcal{F})}^{\mathrm{rat}}$ to $\overline{\mathcal{C}(\gamma\mathcal{F})}^{\mathrm{rat}}$.
In particular: If there is only one $\overline{\mathrm{O}^+(L)}$-orbit of zero-dimensional boundary components, the construction 8.2.1 turns a $\overline{\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}}$-admissible cone decomposition $\Sigma(\mathcal{F}_0)$ for an arbitrary zero-dimensional boundary component $\mathcal{F}_0$ into a Γ-admissible family Σ.

Remark 8.2.3. This is always the case if L is a rescaling of a unimodular lattice L_0 of signature $(2, 8m+2)$: In this case $\mathrm{O}(L) \cong \mathrm{O}(L_0)$ and the uniqueness of the $\mathrm{O}^+(L)$-orbit of zero-dimensional boundary components is just the uniqueness of the indefinite unimodular lattice of signature $(1, 8m+1)$, so there is a non-empty class of examples for this construction.

Admissible families constructed in this way are important in our further work, so we introduce a name for them:

Definition 8.2.4. This Γ-admissible family is called the *symmetric family induced by* $\Sigma(\mathcal{F}_0)$ and the toroidal compactification $\overline{X}_{\Sigma}^{\mathrm{tor}}$ the *symmetric toroidal compactification induced by* $\Sigma(\mathcal{F}_0)$.

Lemma 8.2.2 generalizes to this case:

Proposition 8.2.5. *Let Γ be neat and Σ be the symmetric family induced by $\Sigma(\mathcal{F}_0)$. If every cone in $\Sigma(\mathcal{F}_0)$ is generated by parts of a $\mathbb{Z}$-basis of $\mathcal{U}(\mathcal{F}_0)_{\mathbb{Z}}$, the symmetric toroidal compactification $\overline{X}_{\Sigma}^{tor}$ is smooth.*

We have seen this symmetric construction before:
The globalization of the local construction of the compactification of $\mathcal{E}^{(n)}$ is of this type. Instead of 'constructing locally and translating globally' as we called it in section 3.3, we can take the fan $\Sigma(i\infty)$ and consider the $\Gamma(N)^A$-admissible symmetric family induced by it, since $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on the rational boundary components $\mathbb{P}^1(\mathbb{Q})$ of $\mathbb{H} \times \mathbb{C}^n$: Its toroidal compactification is obviously the same compactification as the one obtained by the construction in section 3.3.
One further comment: If $\Gamma = \widetilde{\mathrm{SO}}^+(L)$ is the discriminant kernel of F, then

$$\widetilde{\mathrm{SO}}^+(K) \cong \overline{\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}} \subseteq \mathrm{Aut}(\mathcal{C}(\mathcal{F}_0))$$

is an arithmetic subgroup with possibly very natural geometric action on $\mathcal{C}(\mathcal{F}_0)$. We will use this in chapter 11 to construct a very natural toroidal compactification for certain lattices.

8.3. Non-neat subgroups and canonical singularities

We give an overview of the singularities of toroidal compactification of non-smooth orthogonal locally symmetric spaces.
In most of the preceding construction of toroidal compactifications we did not assume $\Gamma\backslash\mathcal{D}_L$ to be smooth. Since a toroidal compactification of a locally symmetric space modifies only its boundary, one cannot expect to get smooth toroidal compactifications in general. Actual singularities occur for many of the most naturally appearing objects, namely the locally symmetric spaces associated to unimodular lattices:
Consider the lattice $L = I\!I_{2,10}$. It contains copies of the unimodular negative definite $\mathrm{E}_8(-1)$-lattice and hence its orthogonal group is contained in the discriminant kernel $\Gamma(I\!I_{2,10})$ of $I\!I_{2,10}$, so the latter group is not neat and the associated modular variety $X(L)$ exhibits finite-quotient singularities at possibly every point. In particular, no toroidal compactification of $X(L)$ will be smooth.
We describe the singularities of orthogonal locally symmetric spaces and their toroidal compactifications. A central concept for classifying singularities is the following:

Definition 8.3.1. Let X be a normal complex variety and $\tilde{X}$ any resolution of singularities. The variety X has canonical singularities if and only if on any open set $U \subseteq X$, any pluricanonical form (that is, a section of a multiple of the canonical bundle rK_X) on the smooth part of U extends holomorphically to $\tilde{U} \subseteq \tilde{X}$.

By the work of Gritsenko, Hulek and Sankaran, the only singularities of orthogonal locally symmetric spaces of sufficiently high rank are of this type:

Proposition 8.3.2 ([GHS07b, Corollary 2.16]). *Let L be an even non-degenerate lattice of signature $(2, n)$ with $n \geq 7$ and $\Gamma \subseteq \mathrm{O}(L)$ an arithmetic subgroup. Then: The locally symmetric space $\Gamma\backslash\mathcal{D}_L$ has only canonical singularities.*

The authors build up on this and proved the existence of a toroidal compactification with only canonical singularities for sufficiently high n.
Going through the proofs in [GHS07b], one even gets a criterion for a toroidal compactification of $X = \Gamma\backslash\mathcal{D}_L$ for $\Gamma = \widetilde{\mathrm{O}(L)}$ to have only canonical singularities. Note that for general Γ the argument has to be supplemented by the result of Ma in [Ma18, Appendix], but the statement remains nevertheless true.

Lemma 8.3.3. *If Σ in the preceding case is smooth and $n \geq 9$, the toroidal compactification $\overline{X}_\Sigma^{tor}$ has only canonical singularities.*

Note that smoothness of Σ is not sufficient for the smoothness of $\Gamma\backslash\mathcal{D}_L$, but neatness of Γ together with smoothness of Σ implies the smoothness of $\overline{X}_\Sigma^{\mathrm{tor}}$ by lemma 3.1.7.
In a sense, this shows that the toroidal compactification procedure does not add additional singularities if Σ is smooth.
This finishes this very short chapter on orthogonal toroidal compactifications. The next chapter will describe certain classes of divisors on these compactifications.

9. Divisors in the (compactified) orthogonal case

In this chapter we will describe the geometry of three types of divisors on a toroidal compactification of a locally symmetric space $X = \widetilde{\mathrm{SO}}^+(L)\backslash\mathcal{D}_L$ of orthogonal type: The first two of these are so-called *toroidal boundary divisors* which arise from the compactification procedure, while the third will be the toroidal equivalents of the *Heegner divisors* from definition 7.2.1.
The distinction between the different kinds of divisors also reflects the structure of this chapter: The first section deals with boundary components and divisors of so-called one-dimensional type, while the second section is about those of zero-dimensional type. The third and final section treats Heegner divisors on locally symmetric spaces of orthogonal type in some detail and considers their closure in the Baily-Borel compactification. The closure in general toroidal compactifications is rather complicated: We will consider this only for a certain class of Heegner divisors in a very special toroidal compactification later on in section 11.3.
For this chapter and as always in this part of the thesis, we fix an even non-degenerate lattice L with corresponding symmetric space $\mathcal{D}_L$ and a $\widetilde{\mathrm{SO}}^+(L)$-admissible family Σ. Together, these define a toroidal compactification $\overline{X}^{\mathrm{tor}}_{\Sigma}$ of $\widetilde{\mathrm{SO}}^+(L)\backslash\mathcal{D}_L$. Moreover, we assume $\widetilde{\mathrm{SO}}^+(L)$ to be neat and, for technical reasons, that L is the rescaling of a maximal lattice. We abbreviate $\Gamma = \widetilde{\mathrm{SO}}^+(L)$ in the following.
We start with the classification of the toroidal boundary divisors. By the following result, we can stratify the boundary of $\overline{X}^{\mathrm{tor}}_{\Sigma}$ via the Baily-Borel compactification:

Proposition 9.0.1 ([AMRT10, Chapter III, Proposition 5.3])**.** *There exists a natural surjective morphism $\pi : \overline{X}^{tor}_{\Sigma} \to \overline{X}^{BB}$, inducing the identity morphism on X.*

We can use this to define a stratification of $\overline{X}^{\mathrm{tor}}_{\Sigma}$ indexed by the boundary components of $\overline{X}^{\mathrm{BB}}$:

Definition 9.0.2. Let $F \subseteq \overline{X}^{\mathrm{BB}} \setminus X$ be a connected component of the boundary. The set of points in $\overline{X}^{\mathrm{tor}}_{\Sigma}$ mapped to F under the morphism $\overline{X}^{\mathrm{tor}}_{\Sigma} \to \overline{X}^{\mathrm{BB}}$ is the *F-stratum.* This gives a stratification of $\overline{X}^{\mathrm{tor}}_{\Sigma}$ into locally closed subspaces.

In our case of $\mathcal{D}_L$ for L of signature $(2, n)$, the boundary of the Baily-Borel compactification consists of zero- and one-dimensional cusps. We will call the associated strata of a toroidal compactification *boundary component of zero- resp. one-dimensional type* if the corresponding stratum of $\overline{X}^{\mathrm{BB}}$ is of this dimension. One has to keep in mind that

this name does not hint at the actual dimension of the toroidal strata, since these are all of dimension $n-1 \neq 0,1$ by construction.
The closure $\overline{D} \subseteq \overline{X}_{\Sigma}^{\mathrm{tor}}$ of a toroidal boundary component D of zero- resp. one-dimensional type is a divisor on $\overline{X}_{\Sigma}^{\mathrm{tor}}$ which we will refer to as a *toroidal boundary divisor of zero- resp. one-dimensional type*; the Baily-Borel cusp $F = \pi(D)$ will be called the *underlying cusp* of D.
The description of toroidal boundary divisors decomposes naturally into two parts: The description of the toroidal boundary component and the description of the corresponding divisor obtained by taking closures inside $\overline{X}_{\Sigma}^{\mathrm{tor}}$. As we will see, the first of these depends only on the cone decomposition $\Sigma(F) = \Sigma(\mathcal{F})$ of the underlying cusp $F = \Gamma\mathcal{F}$.
Parts of the results in this chapter have been obtained by Bruinier and Zemel in [BZ19], developed simultaneously to the work presented here. Since the geometric description and treatment of [BZ19] is clearer than the more explicit one of the author, we present their approach.

9.1. Boundary components and divisors of one-dimensional type

We start with the arithmetically richer objects, the toroidal boundary divisor of one-dimensional type.
The structure of the strata of one-dimensional type is more uniform yet more intricate than for the ones of zero-dimensional type: For F_1 a one-dimensional Baily-Borel cusp the cone in $\mathcal{U}(F_1)$ is trivial, i.e.

$$\mathcal{C}(F_1) = \mathbb{R}_{>0}$$

and hence the corresponding cone decomposition is unique, so the toroidal boundary component does not depend on the family Σ of cone decompositions.

Boundary components of one-dimensional type

We recall the local construction of the toroidal compactification at one-dimensional Baily-Borel cusps:
Let F_1 be a one-dimensional Baily-Borel boundary component and $F_0 \subseteq \overline{F_1}$ an adjacent zero-dimensional Baily-Borel boundary component. Henceforth we will denote this situation by

$$F_0 \preceq F_1.$$

We use the notation of section 6.3 for the groups and objects related to F_1.
The way to think of this is the following: The closure of the toroidal boundary components over one-dimensional cusp F_1 is determined by the partial compactifications over the zero-dimensional cusps F_0 with $F_0 \preceq F_1$: Let D be a toroidal boundary component of one-dimensional type. We have

$$\pi\left(\overline{D}\right) \subseteq \overline{\pi\left(D\right)} = \overline{F_1} = F_1 \sqcup \bigsqcup_{F_0 \preceq F_1} F_0,$$

so the boundary parts of $\overline{D}$ lie over the zero-dimensional cusp with $F_0 \preceq F_1$ as stated.
To get a better description of these boundary components we observe that for any rational boundary component $\mathcal{F}$ of $\mathcal{D}$ there is an open neighborhood $N(\mathcal{F})$ of $\mathcal{F} \subseteq \overline{\mathcal{D}}$ which is invariant under the action of the corresponding parabolic $\mathcal{P}(\mathcal{F})_{\mathbb{Z}}$ and on which Γ-equivalence reduces to $\mathcal{P}(\mathcal{F})_{\mathbb{Z}}$-equivalence, cf. [BZ19, Lemma 2.10]. This is equivalent to saying that

$$\mathcal{P}(\mathcal{F})_{\mathbb{Z}} \backslash N(\mathcal{F})$$

is a neighborhood of F in $X = \Gamma \backslash \mathcal{D}_L$. This subset can be chosen by requiring the quadratic form of the imaginary part of the cone variables in the Siegel domain realization of $\mathcal{D}$ with respect to $\mathcal{F}$ to be suitably large.
The corresponding decomposition is $\Sigma(F_1) = \{\{0\}, \mathbb{R}_{>0}\}$. The general construction procedure for partial compactifications of X with respect to F_1 reduces in this case to the situation

$$\begin{aligned}
\mathcal{U}(F_1)_{\mathbb{Z}} \backslash \mathcal{D} &= \mathcal{U}(F_1)_{\mathbb{Z}} \backslash \left\{ (u, \tau, \overrightarrow{z}) \in \mathbb{C} \times \mathbb{H} \times \mathbb{C}^{n-2} : \operatorname{Im}(u) \in \mathbb{R}_{>0} + \frac{1}{\operatorname{Im}(\tau)} q_\Lambda(\overrightarrow{z}) \right\} \\
&= \left\{ (u + \mathbb{Z}, \tau, \overrightarrow{z}) \in \mathbb{C}/\mathbb{Z} \times \mathbb{H} \times \mathbb{C}^{n-2} : \operatorname{Im}(u) \in \mathbb{R}_{>0} + \frac{1}{\operatorname{Im}(\tau)} q_\Lambda(\overrightarrow{z}) \right\} \\
&\subseteq \mathbb{C}/\mathbb{Z} \times \mathbb{H} \times \mathbb{C}^{n-2} \\
&\cong \mathbb{C}^* \times \mathbb{H} \times \mathbb{C}^{n-2} \\
&\subseteq (\mathbb{C}^*)_{\Sigma(F_1)} \times \mathbb{H} \times \mathbb{C}^{n-2} \\
&= \mathbb{C} \times \mathbb{H} \times \mathbb{C}^{n-2}
\end{aligned}$$

cf. definition 3.1.3. The interior of the closure of $\mathcal{U}(F_1)_{\mathbb{Z}} \backslash \mathcal{D}$ in the latter space is

$$\{0\} \times \mathbb{H} \times \mathbb{C}^{n-2}.$$

To get the corresponding toroidal boundary component in $\overline{X}_{\Sigma}^{\text{tor}}$ we still have to take the quotient by the action of the integral parabolic subgroup $\mathcal{P}(\mathcal{F}_1)_{\mathbb{Z}}$ which has a rather delicate inner structure (cf. [Zem20]).
In [Zem20], Zemel determined the structure of this group and the boundary component in great detail. We recall and reformulate the concept of *Kuga-Sato varieties* which already appeared in section 3.3.

Definition 9.1.1. Let Γ_1 be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and consider the modular curve $X(\Gamma_1) = \Gamma_1 \backslash \mathbb{H}$. The universal elliptic curve $\mathcal{E} \to X(\Gamma_1)$ is the universal surface $\mathcal{E}$ with fiber E_τ over $\Gamma_1 \tau$. For a lattice Λ the *open Kuga-Sato variety* $W_{\Gamma_1}^{\Lambda}$ is the variety with fiber $E_\tau \otimes \Lambda$ over $\Gamma_1 \tau$.

This language enables us to state the main result of [Zem20]:

Proposition 9.1.2. *Let F_1 be a one-dimensional boundary component of the Baily-Borel compactification of X. The toroidal boundary component over F_1 is the Kuga-Sato* *variety $W_{\Gamma_1}^{\Lambda}$. Here, Λ is the negative definite sublattice of L defined by the cusp F_1 (resp. the corresponding rational boundary component $\mathcal{F}_1$ as in section 6.3) and Γ_1 is the congruence subgroup of $\mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{Aut}(F_1)$ that arises by intersecting Γ with $G_h(\mathcal{F}_1)$ as in section 6.3.*

Proof. The general form of the toroidal boundary component is described in [Zem20, Theorem 4.5]. We note that, in the local notation, Γ_Λ is finite and neat, so it is trivial and $\iota = 0$ by the assumptions on L. This is what is called the *preliminary version of the boundary component* just before Proposition 4.4. The structure of Γ_1 is described right after Proposition 3.2 in [Zem20]. □

The actual result of [Zem20] is way more general and gives the structure of the canonical toroidal boundary component of $\overline{X}_{\Sigma}^{\mathrm{tor}}$ even for non-neat Γ arising as a discriminant kernel for lattices that are not a rescaling of a maximal lattice. In our case, all of this reduces to the aforementioned result.
Since the open Kuga-Sato variety $W_{\Gamma_1}^{\Lambda}$ is the $\mathrm{rk}(\Lambda)$-fold self-fiber-product of the universal elliptic curve $\mathcal{E} \to \Gamma_1 \backslash \mathbb{H}$ over the modular curve $\Gamma_1 \backslash \mathbb{H}$ we will denote it by $\mathcal{E}^{(\mathrm{rk}(\Lambda))}$ to emphasize the dependence on the rank of Λ. In the case of $\mathrm{sig}(L) = (2, n)$ this implies $\mathrm{rk}(\Lambda) = n - 2$.
As said before, this is only the form of the toroidal boundary component

$$D = \mathcal{E}^{(n-2)}$$

of one-dimensional type, not of the corresponding divisor $\overline{\mathcal{E}^{(n-2)}}$ which is the closure of this boundary component in $\overline{X}_{\Sigma}^{\mathrm{tor}}$.

Boundary divisor of one-dimensional type

The boundary of the divisor $\overline{D}$ is supported on the cusps in the boundary of $\overline{F_1}$. To get a clearer picture of this we will give a more down-to-earth construction of the boundary component than the one provided in proposition 9.1.2 by having a closer look at the gluing procedure described in definition 3.1.5: We recall the equivalence relation responsible for the gluing of the partial compactifications:
For $x \in (\mathcal{U}(\mathcal{F}_i)_{\mathbb{Z}} \backslash \mathcal{D})_{\Sigma(\mathcal{F}_i)}$ and $y \in (\mathcal{U}(\mathcal{F}_j)_{\mathbb{Z}} \backslash \mathcal{D})_{\Sigma(\mathcal{F}_j)}$ we define $x \sim y$ if and only if there exists a rational boundary component $\mathcal{F}$, $\gamma \in \Gamma$ and $z \in (\mathcal{U}(\mathcal{F})_{\mathbb{Z}} \backslash \mathcal{D})_{\Sigma(\mathcal{F})}$ with $\mathcal{F}_i, \gamma\mathcal{F}_j \subseteq \overline{\mathcal{F}}$ and

- z projects to x via the canonical mapping

$$(\mathcal{U}(\mathcal{F})_{\mathbb{Z}} \backslash \mathcal{D})_{\Sigma(\mathcal{F})} \to (\mathcal{U}(\mathcal{F}_i)_{\mathbb{Z}} \backslash \mathcal{D})_{\Sigma(\mathcal{F}_i)}$$

- z projects to γy via the canonical mapping

$$(\mathcal{U}(\mathcal{F})_{\mathbb{Z}} \backslash \mathcal{D})_{\Sigma(\mathcal{F})} \to (\mathcal{U}(\gamma\mathcal{F}_j)_{\mathbb{Z}} \backslash \mathcal{D})_{\Sigma(\gamma\mathcal{F}_j)}\,.$$

We are interested in the partial compactifications getting glued with $(\mathcal{U}(\mathcal{F}_1)_\mathbb{Z}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_1)}$: Going through the possible cases of types of boundary components one-by-one we see that the only relevant gluing is via the map

$$\pi_{\mathcal{F}_1,\mathcal{F}_0} : (\mathcal{U}(\mathcal{F}_1)_\mathbb{Z}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_1)} \to (\mathcal{U}(\mathcal{F}_0)_\mathbb{Z}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_0)}$$

for any $\mathcal{F}_0$ such that $F_0 \preceq F_1$. We fix such $F_0 \preceq F_1$ and can assume the preimages $\mathcal{F}_0, \mathcal{F}_1$ to be compatibly such that

$$\mathcal{U}(\mathcal{F}_0) \supseteq \mathcal{U}(\mathcal{F}_1).$$

The gluing map is given by quotienting modulo

$$\mathcal{U}(\mathcal{F}_0)_\mathbb{Z}/\mathcal{U}(\mathcal{F}_1)_\mathbb{Z} \cong K/\mathbb{Z}\rho_0 \cong \mathbb{Z}^n/\mathbb{Z} \cong \mathbb{Z}^{n-1} \cong \mathbb{Z}\times\mathbb{Z}^{n-2}$$

with K the Lorentzian lattice as in section 8.1 and ρ_0 a primitive generator of the isotropic ray $\mathcal{C}(\mathcal{F}_1)$ defining $\mathcal{F}_1$ in the Siegel domain realization of $\mathcal{D}$ with respect to $\mathcal{F}_0$. This map realizes the quotient of the partial compactification $(\mathcal{U}(\mathcal{F}_1)_\mathbb{Z}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_1)}$ by $\mathcal{U}(F_0)_\mathbb{Z}/\mathcal{U}(F_1)_\mathbb{Z}$ as an open subset of $(U_\mathbb{Z}(F_0)\backslash\mathcal{D})_{\Sigma(F_0)}$. To be more explicit: This is induced by the inclusion

$$\begin{aligned}
(\mathcal{U}(\mathcal{F}_0)_\mathbb{Z}/\mathcal{U}(\mathcal{F}_1)_\mathbb{Z}) \setminus (\mathcal{U}(\mathcal{F}_1)_\mathbb{Z}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_1)} &= (K/\mathbb{Z}\rho_0) \setminus \left(\mathbb{C}\times\mathbb{H}\times\mathbb{C}^{n-2}\right)\\
&= \left(\mathbb{C}\times\mathbb{Z}\backslash\mathbb{H}\times\mathbb{Z}^{n-2}\backslash\mathbb{C}^{n-2}\right)\\
&\cong \mathbb{C}\times\Delta^*\times(\mathbb{C}^*)^{n-2}\\
&\subseteq \left(\mathbb{C}^*\times\mathbb{C}^*\times(\mathbb{C}^*)^{n-2}\right)_{\Sigma(F_0)}\\
&\supseteq (\mathcal{U}(\mathcal{F}_0)_\mathbb{Z}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_0)}.
\end{aligned}$$

The isomorphy here is via a suitable normalized exponential function. One can see that $\mathbb{C}\times\Delta^*\times(\mathbb{C}^*)^{n-2}$ corresponds to the torus orbit of the isotropic cone $\mathcal{C}(\mathcal{F}_1) = \mathbb{R}_{>0}\omega$ with $\mathbb{R}\omega+\mathbb{R}\tau = \mathcal{F}_1 \supseteq \mathcal{F}_0 = \mathbb{R}\tau$.
This shows how to obtain the closure of the toroidal boundary component over F_1 near the cusp F_0: It is the image of

$$\partial\left(\overline{\pi_{\mathcal{F}_1,\mathcal{F}_0}\left((\mathcal{U}(\mathcal{F}_1)_\mathbb{Z}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_1)}\right)}\right) \subseteq (\mathcal{U}(\mathcal{F}_0)_\mathbb{Z}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_0)}$$

under Γ-equivalence. The action of Γ on this set is of course only as

$$[(\mathcal{P}(\mathcal{F}_0)_\mathbb{Z}\cap\mathcal{P}(\mathcal{F}_1)_\mathbb{Z})\,/\,(\mathcal{U}(\mathcal{F}_1)_\mathbb{Z})]\,/\,[(\mathcal{U}(\mathcal{F}_0)_\mathbb{Z}/\mathcal{U}(\mathcal{F}_1)_\mathbb{Z})] \cong (\mathcal{P}(\mathcal{F}_0)_\mathbb{Z}\cap\mathcal{P}(\mathcal{F}_1)_\mathbb{Z})\,/\,(\mathcal{U}(\mathcal{F}_0)_\mathbb{Z})$$

as all of this is happening in a suitable neighborhood of $F_0\cup F_1$, e.g. an intersection of the sets $\mathcal{N}(F_0)$ and $\mathcal{N}(F_1)$ from above (so we have to consider only the action of $\mathcal{P}(\mathcal{F}_0)_\mathbb{Z}\cap\mathcal{P}(\mathcal{F}_1)_\mathbb{Z}$), and $\pi_{\mathcal{F}_1,\mathcal{F}_0}$ already takes the quotient by $\mathcal{U}(\mathcal{F}_0)_\mathbb{Z}/\mathcal{U}(\mathcal{F}_1)_\mathbb{Z}$.
We recall some notions from the theory of toric varieties: Let $X_{L_0,\Sigma}$ resp. $X_{L_0,\tau}$ for the toric variety of a fan Σ_0 with respect to a lattice L_0 resp. the affine toric variety with respect to the cone σ_0.

For $L_0 = K$, $\Sigma_0 = \Sigma(\mathcal{F}_0)$ and $\sigma_0 = \mathcal{C}(\mathcal{F}_1) \in \Sigma(\mathcal{F}_0)$ we get

$$X_{K,\Sigma(\mathcal{F}_0)} = \left(\mathbb{C}^* \times \Delta^* \times (\mathbb{C}^*)^{n-2}\right)_{\Sigma(\mathcal{F}_0)} \text{ and } X_{K,\mathcal{C}(\mathcal{F}_1)} = (\mathbb{C}^*)_{\Sigma(F_1)} \times \Delta^* \times (\mathbb{C}^*)^{n-2}.$$

From the theory of toric varieties (lemma 2.1.13) we know that

$$\overline{X_{K,\mathcal{C}(\mathcal{F}_1)}} = \bigcup_{\mathcal{C}(\mathcal{F}_1) \preceq \sigma \in \Sigma(\mathcal{F}_0)} X_{K,\sigma}$$

which itself can be thought of as a toric variety for a lattice of lower rank. We recall these statements here in our special case:

Proposition 9.1.3. *The closure $\overline{X_{K,\mathcal{C}(\mathcal{F}_1)}}$ of $X_{K,\mathcal{C}(\mathcal{F}_1)}$ is isomorphic to the toric variety*

$$X_{N,\mathrm{Star}_{\Sigma(\mathcal{F}_0)}(\mathcal{C}(\mathcal{F}_1))}$$

with $N = \mathcal{U}(\mathcal{F}_0)_{\mathbb{Z}}/\mathcal{U}(\mathcal{F}_1)_{\mathbb{Z}} = K/\mathbb{Z}\rho_0$ and $\mathrm{Star}_{\Sigma(\mathcal{F}_0)}(\mathcal{C}(\mathcal{F}_1))$ the star of $\mathcal{C}(\mathcal{F}_1)$ in $\Sigma(\mathcal{F}_0)$, a rational polyhedral cone decomposition of the cone

$$\mathcal{C}(\mathcal{F}_0)/\mathcal{U}(\mathcal{F}_1)_{\mathbb{Z}} \cong \left\{(x_1, \overrightarrow{x_4}) \in \mathbb{R} \times \mathbb{R}^{n-2} : x_1 > 0\right\} \subseteq \mathbb{R}^{n-1} = \mathcal{U}(\mathcal{F}_0)/\mathcal{U}(\mathcal{F}_1).$$

Proof. This is mostly lemma 2.1.13 applied to this concrete situation. The form of the quotient cone can be derived from the action of $\mathcal{U}(\mathcal{F}_1)_{\mathbb{Z}}$ as translation on the second coordinate on the cone $\mathcal{C}(\mathcal{F}_0)$ in the Siegel domain realization with respect to $\mathcal{F}_0$: With respect to the basis $(e, f, e', f', \ldots)$ the cone $\mathcal{C}(\mathcal{F}_0)$ is given by

$$\mathcal{C}(\mathcal{F}_0) = \{(x_1, x_3, \overrightarrow{x_4}) \in \mathbb{R}^n | \, x_1 > 0, x_1x_3 - q_\Lambda\left(\overrightarrow{x_4}\right) > 0\}$$

and $u \in \mathcal{U}(\mathcal{F}_1) \cong \mathbb{R}$ acts on this by $x_3 \mapsto x_3 + u$, hence the quotient cone is isomorphic to

$$\{(x_1, \overrightarrow{x_4}) \in \mathbb{R}^n | x_1 > 0\} \cong \mathbb{R}_+ \times \mathbb{R}^{n-2},$$

which proves the claims. $\square$

The closure of the toroidal boundary component over F_1 in a neighborhood of $F_0 \subseteq \overline{F_1}$ is now given as the image of $X_{N,\mathrm{Star}_{\Sigma(\mathcal{F}_0)}(\mathcal{C}(\mathcal{F}_1))}$ under the action of

$$\left(\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}} \cap \mathcal{P}(\mathcal{F}_1)_{\mathbb{Z}}\right)/\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}} \cong \Lambda.$$

Doing this construction for any zero-dimensional boundary component adjacent to F_1 yields a complete picture of the toroidal boundary component over F_1. In particular, the boundary of the closure of the toroidal boundary *component* consists of toric varieties corresponding to cones in the cone decomposition Σ_{F_0} containing the ray defining F_1. The attentive reader is reminded of the construction of the toroidal compactifications of the fiber powers of the universal elliptic curve over the modular curve (which we now call *Kuga-Sato variety* in view of proposition 9.1.2). This is no coincidence:

Proposition 9.1.4. *The toroidal divisor of one-dimensional type over the Baily-Borel cusp $F_1 = \Gamma_1\backslash\mathbb{H}$ is isomorphic to a toroidal compactification of the Kuga-Sato variety $K^{\Lambda}_{\Gamma_1}$ as in section 3.3.*
This toroidal compactification of $K^{\Lambda}_{\Gamma_1}$ is defined by the collection of fans induced by the Γ-admissible collection Σ in the following way: For any cusp $F_0 \preceq F_1$ the cone decomposition is given by the star $\mathrm{Star}_{\Sigma(\mathcal{F}_0)}(\mathcal{C}(\mathcal{F}_1))$ of $\mathcal{C}(\mathcal{F}_1)$ in $\Sigma(\mathcal{F}_0)$.

This approach gives a rich resource for compactifications of Kuga-Sato varieties resp. fiber products of universal elliptic curves. The recipe is as follows: Realize them as the toroidal boundary components over one-dimensional Baily-Borel cusps of a orthogonal Shimura variety, compactify the latter toroidally and consider the closure of the boundary component in this compactification.
To the best of the author's understanding, this approach is the main tool in the work [Lan12b] of Lan.
For later combinatorial considerations it is useful to know the number of these divisors. This is easy:

Lemma 9.1.5. *The number of boundary divisor of one-dimensional type of $\overline{X}^{rat}_{\Sigma}$ is equal to the number ν_1 of one-dimensional Baily-Borel cusps of X.*

9.2. Boundary components and divisors of zero-dimensional type

We turn to the second type of toroidal boundary components.
The toroidal boundary components of zero-dimensional type are less uniform than those of one-dimensional type as they actually do depend on the Γ-admissible family Σ. Similar to the case of toroidal boundary components of one-dimensional type we will give a concrete (but less detailed) construction:
Fix a zero-dimensional F_0 with preimage $\mathcal{F}_0$ and cone decomposition $\Sigma(\mathcal{F}_0)$. We use again the notation of section 6.3. The partial compactification with respect to this rational boundary component is

$$\begin{array}{rcl} \mathcal{U}(\mathcal{F}_0)_{\mathbb{Z}}\backslash\mathcal{D} & \subseteq & \mathcal{U}(\mathcal{F}_0)_{\mathbb{Z}}\backslash\mathcal{U}(\mathcal{F}_0)_{\mathbb{C}} \\ & \subseteq & K\backslash\mathbb{C}^n \\ & \cong & (\mathbb{C}^*)^n \\ & \subseteq & ((\mathbb{C}^*)^n)_{\Sigma(\mathcal{F}_0)} \\ & \supseteq & (\mathcal{U}(\mathcal{F}_0)_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_0)} \end{array}$$

cf. definition 3.1.3, with the Lorentzian lattice K as in the decomposition of L with respect to $\mathcal{F}_0$ in section 8.1.
The partial compactification with respect to $\mathcal{F}_0$ is the interior of the closure of $\mathcal{U}(\mathcal{F}_0)_{\mathbb{Z}}\backslash\mathcal{D}$ in the toric variety $((\mathbb{C}^*)^n)_{\Sigma(\mathcal{F}_0)}$. This partial compactification contains a lot of divisors:

For each ray $\rho_i \in \Sigma(\mathcal{F}_0)$ we have the closure D_{ρ_i} of the torus orbit $O_{\Sigma(\mathcal{F}_0)}(\rho_i)$. To get the corresponding divisors on $\overline{X}_\Sigma^{\mathrm{tor}}$ we again have to take into account the gluing via the equivalence relation described in definition 3.1.5.
This time there is more than one type of relevant gluing maps:
First of all, again, the maps

$$\pi_{\mathcal{F}_1,\mathcal{F}_0} : (\mathcal{U}(\mathcal{F}_1)_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_1)} \to (\mathcal{U}(\mathcal{F}_0)_{\mathbb{Z}}\backslash\mathcal{D})_{\Sigma(\mathcal{F}_0)}$$

for $\mathcal{F}_0$ with $F_0 \prec F_1$ show that the toric boundary components $O_{\Sigma(\mathcal{F}_0)}(\rho)$ for isotropic ρ (which are the image of $\mathcal{C}(\mathcal{F}_1)$ in $\mathcal{C}(\mathcal{F}_0)$) correspond to the toroidal boundary components of one-dimensional type. These have been treated in the last section, so we exclude them from this consideration of the toroidal divisors of zero-dimensional type.
The other type of relevant gluing maps is the identification via the maps $\pi_{\mathcal{F}_0,\mathcal{F}_0} = \mathrm{id}$ and the action of Γ on these partial compactifications: We recall that the realization as a Siegel domain of the third kind with respect to $\mathcal{F}_0$ is as the tube domain

$$\mathcal{D} \cong \mathbb{R}^n + i\mathcal{C}(\mathcal{F}_0).$$

For determining the boundary divisors and components it suffices to view the action of Γ in the neighborhood $\mathcal{N}(\mathcal{F}_0)$ where Γ acts only via $\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}$.
Here, $\gamma \in \Gamma$ acts by translation via $u(\gamma) \in K$ via the projection

$$\mathcal{P}(\mathcal{F}_0)_Z \to \mathcal{U}(\mathcal{F}_0)_{\mathbb{Z}} \cong K, \gamma \mapsto u(\gamma)$$

on the real part and by $r(\gamma) \in \widetilde{\mathrm{SO}}^+(K)$ via

$$\mathcal{P}(\mathcal{F}_0)_Z \to G_l(\mathcal{F}_0)_{\mathbb{Z}} \cong \widetilde{\mathrm{SO}}^+(K), \gamma \mapsto r(\gamma)$$

on the real part and on the cone $\mathcal{C}(\mathcal{F}_0)$ in the imaginary part.
This latter part shows how the gluing of divisor is achieved: The action of $\gamma \in G_l(\mathcal{F}_0)_{\mathbb{Z}}$ identifies the torus orbits $O(\sigma)$ and $O(\gamma\sigma)$ for all $\sigma \in \Sigma(\mathcal{F}_0)$, so the toroidal boundary components of zero-dimensional type over F_0 can be represented by $O_{\Sigma(\mathcal{F}_0)}(\rho)$ for ρ running through a system of $G_l(\mathcal{F}_0)_{\mathbb{Z}} = \widetilde{\mathrm{SO}}^+(K)$-representatives of non-isotropic rays in the decomposition $\Sigma(\mathcal{F}_0)$ of $\overline{\mathcal{C}(\mathcal{F}_0)}^{\mathrm{rat}}$.
By the orbit-cone correspondence we can also describe the closure $\overline{D}$ of such a boundary component $D = O(\rho)$ as the quotient of the toric variety $X_{\mathrm{Star}_{\Sigma(\mathcal{F}_0)}(\rho)}$ corresponding to the star $\mathrm{Star}_{\Sigma(\mathcal{F}_0)}(\rho)$ of $\rho \in \Sigma(\mathcal{F}_0)$ by $\widetilde{\mathrm{SO}}^+(K)$.
We have thus proved:

Proposition 9.2.1. *The toroidal boundary divisors of zero-dimensional type over F_0 are given by the quotient*

$$X_{\mathrm{Star}_{\Sigma(\mathcal{F}_0)}(\rho)}/\widetilde{\mathrm{SO}}^+(K)$$

of a toric variety, where ρ runs through a system of $\widetilde{\mathrm{SO}}^+(K)$-representatives of non-isotropic rays and K denotes the signature-$(1, n-1)$ sublattice of L obtained by a choice of $\mathcal{F}_0$ for $F_0 = \Gamma\mathcal{F}_0$ as in section 6.3.

This is in accordance with the independently obtained description in [BZ19], right after Theorem 3.25.

Remark 9.2.2. The action of $G_l(\mathcal{F}_0)_\mathbb{Z} = \widetilde{\mathrm{SO}}^+(K)$ may glue the toric variety $X_{\mathrm{Star}_{\Sigma(\mathcal{F}_0)}(\rho)}$ with itself and cause self-intersection of the corresponding toroidal divisor: Proper self-gluing of a divisor

$$X_{\mathrm{Star}_{\Sigma(\mathcal{F}_0)}(\rho)}$$

for some non-isotropic ray ρ under the action of $\widetilde{\mathrm{SO}}^+(K)$ means that there is a automorphism $\gamma \in \widetilde{\mathrm{SO}}^+(K)$ and $\sigma \in \Sigma(\mathcal{F}_0)$ with $\sigma \neq \gamma\sigma$ such that $\sigma \cap \gamma\sigma \supseteq \rho$, which is exactly the converse of the condition in lemma 3.1.10 that ensures the boundary divisor of the toroidal compactification being simple normal crossing; we see now the deeper reason for this behavior.

The following is immediate with regard to proposition 2.1.11 and the preceding discussion:

Lemma 9.2.3. *If a Γ-admissible family Σ satisfies the condition in lemma 3.1.10, then every toroidal boundary divisor of zero-dimensional type $\overline{D}$ is a toric variety of the form*

$$X_{\mathrm{Star}_{\Sigma(\mathcal{F}_0)}(\rho)}$$

with ρ a non-isotropic ray in Σ_{F_0}.

This is a variant of the corrected version of [YZ14, Theorem 2.22] for toroidal compactifications of locally symmetric spaces for lattices of signature $(2, n)$: The authors there consider toroidal compactifications of Siegel varieties and characterize those with simple normal crossing divisors by introducing the notions of Γ-*separability* and *geometric* Γ-*fineness* and prove the equivalence with the simple normal crossing property. However, their definition of Γ-separability and the subsequent reasoning is flawed as we explained in the proof of lemma 3.1.10 and the following remark.

9.3. Heegner divisors and embedded Shimura varieties

There is a third class of divisors on Shimura varieties of orthogonal type and their compactifications, the so-called *Heegner divisors.* We encountered variants of these divisors before in definition 7.2.1: These were constructed by the embedding of lower dimensional symmetric spaces into $\mathcal{D}$.
Having mainly taken the point of view of locally symmetric spaces in the meantime, we now introduce Heegner divisors on these space: We denote $\Gamma = \widetilde{\mathrm{SO}}^+(L)$ and by $\overline{X}^{\mathrm{tor}}_\Sigma$, as always, a toroidal compactification of the locally symmetric space $X = X(\Gamma) = \Gamma\backslash\mathcal{D}$.
The notion of Heegner divisors on the symmetric space $\mathcal{D}$ gives rise to divisor on X as follows:

Definition 9.3.1. Let $[\beta] = \beta + L \in L'/L$ be the class of a primitive vector λ of positive norm $q(\lambda) = m > 0$. The image of the Heegner divisor

$$\sum_{\substack{\lambda \in \beta + L \\ q(\lambda) = m}} \lambda^{\perp}$$

in X under $\mathcal{D} \to \Gamma \backslash \mathcal{D} = X$ is a divisor on X, which we will refer to by the same name and notation. The closure of $H(\beta, m)$ in the Baily-Borel compactification $\overline{X}^{\mathrm{BB}}$ resp. any toroidal compactification $\overline{X}^{\mathrm{tor}}_{\Sigma}$ of X is again a divisor on the respective space and will be denoted by $\overline{H(\beta, m)}$ since the surrounding space will be clear from context.

In general, a Heegner divisor $H(\beta, m)$ on X is reducible with the irreducible components given by $\Gamma \backslash \lambda^{\perp}$ where λ runs through a system of representatives for the orbits of Γ on the set of $\lambda \in \beta + L$ with $q(\lambda) = m$. If L splits two hyperbolic planes, the Eichler criterion shows that there is only one Γ-orbit and $H(\beta, m)$ is irreducible itself.
There is a lattice-theoretic way of describing these irreducible components that shows that these components themselves carry geometric structure:
For $\lambda \in L$ with $q(\lambda) < 0$ consider

$$K_{\lambda} = \{z \in L | \, (z, \lambda) = 0\},$$

the orthogonal complement of λ in L (not to be confused with the lattice K arising by the choice of a zero-dimensional rational boundary component $\mathcal{F}_0$). This is a primitive even integral lattice and has signature $(2, n-1)$. We can form its symmetric space

$$\mathcal{D}_{K_{\lambda}} \cong \mathcal{D}_L \cap \lambda^{\perp}$$

as well as its discriminant kernel $\Gamma(K_{\lambda})$ in the same manner as for L (cf. section 6.1). By the abstract properties of the discriminant kernel we have

$$\Gamma(K_{\lambda}) \subset \Gamma(L) := \Gamma.$$

The irreducible components of the divisor $H(\beta, m) \subseteq \mathcal{D}_L$ are obviously of the form $\mathcal{D}_{K_{\lambda}}$ for suitable $\lambda \in L$. The interesting point is that the images of these symmetric spaces in $X = \Gamma \backslash \mathcal{D}_L$ are natural locally symmetric spaces again.
This can be seen by a combination of results of Deligne, Gritsenko-Hulek-Sankaran and Jaffee as follows:

Proposition 9.3.2. *Let L be an even lattice of signature $(2, n)$ such that its discriminant kernel $\Gamma \subseteq \mathrm{O}^+(L)$ is a neat congruence subgroup of $\mathrm{SO}^+(L)$. For primitive $\lambda \in L$ with $|q_L(\lambda)| \neq 4$ the diagram*

$$\begin{array}{ccc} \mathcal{D}_{K_{\lambda}} & \hookrightarrow & \mathcal{D}_L \\ \downarrow{\scriptstyle \pi_{\Gamma(K_{\lambda})}} & & \downarrow{\scriptstyle \pi_{\Gamma(L)}} \\ \Gamma(K_{\lambda}) \backslash \mathcal{D}_{K_{\lambda}} & \hookrightarrow & \Gamma(L) \backslash \mathcal{D}_L \end{array}$$

commutes and has the following further properties: The lower horizontal map is the reduction by $\Gamma(L)/\Gamma(K_{\lambda})$ and it is a closed embedding. In particular $\Gamma(K_{\lambda}) \backslash \mathcal{D}_{K_{\lambda}}$ is an embedded sub-Shimura variety of orthogonal type and signature $(2, n-1)$ in $\Gamma(L) \backslash \mathcal{D}_L$.

Even if the above map is not a closed embedding, the construction yields a (singular) divisor on $\Gamma(L)\backslash\mathcal{D}_L$. The rich supply of divisors available in this way is exploited by Kudla in [Kud97]. These objects will also be of great importance to us in the subsequent chapters. Following Kudla we will call them *special divisors*:

Definition 9.3.3. For primitive $\lambda \in L$ the divisor

$$X(K_\lambda) = \Gamma(K_\lambda)\backslash\mathcal{D}_{K_\lambda} \subseteq X(L)$$

with $K_\lambda = \lambda^\perp \subseteq L$ is called a *special divisor.*

Remark 9.3.4. Recalling the language and notion of section 1.2 about Shimura varieties, this name fits well into context: These divisors are, considered as sub-locally symmetric spaces of $X(L)$ exactly the special divisors in the Shimura variety sense. We will sometimes refer to these as *embedded Shimura varieties* or *sub-Shimura varieties.*

We give a full proof of proposition 9.3.2 for the sake of completeness. The heart of the argument is the elegant *Jaffee lemma*, see [MR80, Proposition 2.2].

Lemma 9.3.5. *Let $\sigma = \sigma_\lambda$ be the reflection in $\lambda^\perp$. Suppose Γ acts freely on $\mathcal{D}_L$ with $\sigma\Gamma\sigma = \Gamma$ and denote by Γ_λ the centralizer of σ in Γ. Then the reduction map*

$$\Gamma_\lambda\backslash\mathcal{D}_{K_\lambda} \to \Gamma\backslash\mathcal{D}_L = X(L)$$

is injective.

Proof. Let $x, y \in \mathcal{D}_{K_\lambda}$ with $\gamma x = y$ for some $\gamma \in \Gamma$ and define $\nu = \sigma\gamma\sigma\gamma^{-1}$. We have $\sigma\gamma\sigma \in \Gamma$ by assumption and hence $\nu \in \Gamma$. Since σ leaves $\mathcal{D}_{K_\lambda}$ invariant, we see

$$\nu y = \sigma\gamma\sigma x = \sigma\gamma x = \sigma y = y,$$

so $\nu \in \mathrm{Stab}_\Gamma(y)$, which is trivial as Γ acts freely; hence

$$\sigma\gamma\sigma = \gamma \in \Gamma_\lambda$$

and the reduction map is seen to be injective. □

We are now able to prove proposition 9.3.2:

Proof. As before, let $x, y \in \mathcal{D}_{K_\lambda}$ with $\gamma x = y$ for some $\gamma \in \Gamma$, then we can apply lemma 9.3.5: As the intersection

$$\Gamma = \widetilde{\mathrm{SO}}^+(L) = \widetilde{\mathrm{O}}^+(L) \cap \mathrm{SO}^+$$

of two normal subgroups of $\mathrm{O}^+(L)$, the subgroup Γ is normal, so any element of $\mathrm{O}^+(L)$ including all the reflections are in its normalizer; moreover, neatness implies free action. This shows that $\gamma \in \Gamma_\lambda$. Since

$$(\gamma x, \lambda) = (\sigma\gamma\sigma x, \lambda) = (\gamma x, \sigma\lambda) = -(\gamma x, \lambda)$$

for all $x \in \lambda^\perp$,

$$\Gamma_\lambda \subseteq \{\phi \in \Gamma(L) |\, \phi(K_\lambda) = K_\lambda\} =: \Gamma_{K_\lambda}$$

and the proof of [GHS08, Proposition 2.3] shows that $\Gamma_{K_\lambda} = \Gamma(K_\lambda)$, the discriminant kernel of K_λ. Altogether this shows that

$$\Gamma(K_\lambda)\backslash\mathcal{D}_{K_\lambda} \to \Gamma(L)\backslash\mathcal{D}_L$$

is injective and [Del71b, Proposition 1.15] tells us that it is a closed immersion in this case. $\square$

The relation $\Gamma_\lambda \subseteq \Gamma_{K_\lambda}$ can be strengthened to an equality by the following consideration: Let $\gamma \in \Gamma_{K_\lambda}$. For $x \in \lambda^\perp$ we have

$$0 = \langle x, \lambda\rangle = \langle \gamma x, \gamma\lambda\rangle = \langle x', \gamma\lambda\rangle.$$

As the restriction of γ to $\lambda^\perp$ is an automorphism, we see $\gamma\lambda \in \left(\lambda^\perp\right)^\perp = \mathbb{R}\lambda$ and finally $\gamma\lambda = \lambda$ since $\gamma \in \mathrm{SO}^+(L)$, so $\gamma \in \Gamma_\lambda$.
Due to this fact we will use the notations Γ_λ and Γ_{K_λ} interchangeably.

Remark 9.3.6. One is tempted to apply the argument of proposition 9.3.2 inductively. Indeed, given lattices

$$L_0 \subseteq L_1 \subseteq L_2$$

cut out by suitable vectors $\lambda_i \in L_i$, i.e.

$$L_1 = L_2 \cap \lambda_2^\perp \text{ and } L_0 = L_1 \cap \lambda_1^\perp = L_2 \cap \lambda_2^\perp \cap \lambda_1^\perp$$

the diagram

$$\begin{array}{ccccc}
\mathcal{D}_{L_0} & \xhookrightarrow{f_{0,1}} & \mathcal{D}_{L_1} & \xhookrightarrow{f_{1,2}} & \mathcal{D}_{L_2} \\
\downarrow{\scriptstyle \pi_{\Gamma(L_0)}} & & \downarrow{\scriptstyle \pi_{\Gamma(L_1)}} & & \downarrow{\scriptstyle \pi_{\Gamma(L_2)}} \\
\Gamma(L_0)\backslash\mathcal{D}_{L_0} & \xhookrightarrow{g_{0,1}} & \Gamma(L_1)\backslash\mathcal{D}_{L_1} & \xhookrightarrow{g_{1,2}} & \Gamma(L_2)\backslash\mathcal{D}_{L_2}
\end{array}$$

commutes and the implied morphism

$$g_{1,2} \circ g_{0,1} : \Gamma(L_0)\backslash\mathcal{D}_{L_0} \hookrightarrow \Gamma(L_2)\backslash\mathcal{D}_{L_2}$$

is a closed immersion; analogously, this works as well for longer chains of lattices.
Note that this does not imply a similar result for the intersection products of two distinct special cycles: While for two mutually orthogonal vectors $\lambda_1, \lambda_2 \in L$ the intersection

$$L_0 = L_1 \cap L_2 \subseteq L_1, L_2 \subseteq L$$

gives rise to an embedded Shimura variety $X(L_0)$ of $X(L)$ as described before, this is *not* necessarily the same as the intersection product $X(L_1) \cdot X(L_2)$ of the special cycles of $X(L)$: If $\gamma\lambda_1 = \pm\lambda_2$ for some $\gamma \in \Gamma$, these cycles share a common component and the intersection is not transversal. An exact treatment of the general case can be found in [Kud19] or later on in proposition 12.2.13.

The relation between special divisors and Heegner divisors is more complicated than it initially appears: We assume for the rest of this chapter that

$$X(K_\lambda) \to X(L)$$

is a closed embedding.
The special divisor corresponding to K_λ is an irreducible component of the Heegner divisor $H(0+L, q(\lambda))$ in $\Gamma(L)\backslash\mathcal{D}_L$. Since the orthogonal complement is invariant under scaling, any multiple $s\lambda \in L'$ gives rise to the same lattice $K_\lambda = K_{s\lambda}$ and associated symmetric subspace $\mathcal{D}_{K_\lambda} = \mathcal{D}_{K_{s\lambda}}$. Therefore, the special divisor $\Gamma\backslash\mathcal{D}_{K_\lambda}$ is also an irreducible component of $H(0+L, s^2 q(\lambda))$ and the allocation between special divisor and Heegner divisors is not injective.
However, for lattices without vectors of norm ± 4, it is surjective:

Lemma 9.3.7. *Let $\beta + L \in L'/L$ and $m = q(\beta) < 0$. Let $H(\beta, m)$ be the corresponding Heegner divisor on X and E one of its irreducible components. Then there exists a primitive $\mu \in L$ with*

$$E \cong \Gamma(K_\mu)\backslash\mathcal{D}_{K_\mu}.$$

Proof. The irreducible component E is given by $\Gamma\backslash\lambda^\perp$ for some representative $\lambda \in L'$ of the Γ-orbit of elements in $\beta + L$ of norm m. Just choose $\mu \in L$ as a primitive generator of $\mathbb{Q}\lambda \cap L$, then $\lambda^\perp = \mu^\perp$ and the remaining claims follow from proposition 9.3.2. □

Note that for $\beta \notin L$, the norm of μ is $q(\mu) = q(l\lambda) = l^2 m \neq m$ for some $l \in \mathbb{Z}_{>1}$.
As remarked in the beginning of this section one can consider the closure of this divisor in the Baily-Borel compactification; this was done by Attwell-Duval. The main result of this consideration is:

Proposition 9.3.8 ([AD15, Lemma 7.15])**.** *The closure of $\lambda^\perp$ in $\overline{\mathcal{D}_L}^{rat}$ contains the rational boundary components corresponding to isotropic lines and planes that are perpendicular to λ. The image of this closure under the projection map $\pi : \overline{\mathcal{D}_L}^{rat} \to \overline{\Gamma\backslash\mathcal{D}_L}^{BB}$ is $\overline{\Gamma(K_\lambda)\backslash\mathcal{D}_{K_\lambda}}$.*

In other words: The Baily-Borel compactification $\overline{X(K_\lambda)}^{\mathrm{BB}}$ of $X(K_\lambda)$ is the same as the closure of $X(K_\lambda) \subseteq X(L)$ in the Baily-Borel compactification $\overline{X(L)}^{\mathrm{BB}}$; Baily-Borel compactification and embeddings commute.
This is a concrete description of the abstract statement in lemma 3.2.1 about the morphism ψ_{BB}.
Note that this does not imply that any Baily-Borel cusp of $X(L)$ induces a Baily-Borel cusp of $X(K_\lambda)$ or, if it does, that these cusps are of the same dimension.
We will be far more interested in the closure of $X(K_\lambda)$ in a given toroidal compactification $\overline{X(L)}^{\mathrm{tor}}_\Sigma$ of $X(L)$. In general, this is rather involved, but for special combinations of toroidal compactifications and special divisors there is more to be said. We will come back to this in section 11.3.
This concludes this chapter about Heegner divisors and toroidal boundary divisors. In the next chapter we will return to the question of dimension formulas as in chapter 4 and specialize it to orthogonal modular forms.

10. Dimension formulas for orthogonal locally symmetric spaces

We will specialize the general theory of dimension formulas from chapter 4 to the case of orthogonal locally symmetric spaces and describe the canonical part of the dimension formula in greater detail.
As always, let L be an even non-degenerate lattice of signature $(2, n)$ and $\Gamma \subseteq \mathrm{O}^+(L)$. We consider the locally symmetric space $X = X(\Gamma) = \Gamma\backslash\mathcal{D}_L$. For this section we fix an arbitrary smooth projective toroidal compactification $\overline{X}$ whose boundary divisor is a simple normal crossing divisor. These exist by chapter 3.
The first section of this chapter will describe the canonical part of the general dimension formula of theorem 4.2.5 in greater detail and relate it to orthogonal modular forms; the second section explains some peculiarities of the compactification-dependent error term in the orthogonal case.

10.1. Canonical dimension formula

As the title of this thesis suggests, we would like to compute the dimension of the space $S_k(\Gamma)$ of orthogonal cusp forms of given weight k with level Γ, preferably by the methods outlined in chapter 4, so we have to identify orthogonal cusp forms with general automorphic forms as in section 1.2:
In the setting there, consider the case of $\mathcal{D} = \mathcal{D}_L$ and $\rho = 0$ trivial. A holomorphic ρ-automorphic form of weight k and level Γ in this sense is nothing but a Γ-invariant function $\mathcal{D} \to \mathbb{C}$, satisfying certain growth conditions and inducing a holomorphic section of $E_0 = \Omega_X^{n\,\otimes k}$. This corresponds exactly to the characterization of orthogonal modular forms of weight nk as holomorphic sections of the same bundle in proposition 7.1.3.
We combine this with lemma 4.2.4 formulate it as a lemma:

Lemma 10.1.1. *Let L be an even lattice of signature $(2, n)$, $\Gamma \subseteq \mathrm{O}(L)$ a neat arithmetic subgroup and $\overline{X} = \overline{X}_\Sigma^{tor}$ a smooth toroidal compactification of $\Gamma\backslash\mathcal{D}_L$. We have vector space isomorphisms*

$$
\begin{aligned}
& \{\textit{orthogonal modular forms of weight } nk \textit{ with respect to } \Gamma \textit{ on } \mathcal{D}_L\} \\
\cong\ & \left\{0\textit{-automorphic forms with respect to } E_0 = \Omega_D^{\otimes k}\right\} \\
\cong\ & \left\{\textit{global sections of } \overline{E} = \Omega_{\overline{X}}^n(\log\Delta)^{\otimes k}\right\}
\end{aligned}
$$

In this picture, orthogonal cusp forms of weight nk with respect to Γ correspond to global sections of $\Omega^n_{\overline{X}}(\log \Delta)^{\otimes(k-1)} \otimes \Omega^n_{\overline{X}}$, so

$$S_{nk}(\Gamma) \cong H^0\left(\Omega^n_{\overline{X}}(\log \Delta)^{\otimes(k-1)} \otimes \Omega^n_{\overline{X}}\right).$$

For a different take on the relation between these objects compare with the treatment in [GHS07a, Section 1].

Remark 10.1.2. The difference in the weights of orthogonal modular forms is an unfortunate but inevitable feature of this theory. The weights as sections of line bundles are usually called *geometric* while the ones from definition 7.1.1 are *arithmetic*. Note that we have

$$S_{nk}(\Gamma) = \mathcal{S}_k(\Gamma).$$

We state the result of theorem 4.2.5 explicitly for orthogonal cusp forms, noting that the dimension of the boundary of the Baily-Borel compactification is $\dim \partial \overline{X}^{\mathrm{BB}} = 1$:

Theorem 10.1.3. *For $k \geq 2$ we have*

$$\dim(S_{nk}(\Gamma)) = \mathrm{Vol}_{\mathrm{HM}}(\Gamma)\mathcal{P}(k-1) + E(k)$$

with a polynomial $E(k)$ of degree 1 and the Hilbert polynomial

$$\mathcal{P}(k) = \chi\left(\left(\Omega^n_{\check{\mathcal{D}}}\right)^{-k}\right)$$

of the compact dual $\check{\mathcal{D}} = \mathrm{SO}(2+n)/\left(\mathrm{SO}(2) \times \mathrm{SO}(n)\right)$ of $\mathcal{D}$.

Note that due to Hirzebruch-Mumford proportionality of theorem 4.2.1 the first term is independent of the choice of the toroidal compactifications.
The exact geometry of the boundary of the toroidal compactification will only come into play during the description and computation of the error term $E(k)$, even though the value of $E(k)$ cannot depend on this choice.
This result gives a roadmap to the computation of dimension formulas. It consists of the following three steps, of which the last one is the most demanding:

1) Compute the Hirzebruch-Mumford volume $\mathrm{Vol}_{\mathrm{HM}}(\Gamma) \in \mathbb{Q}$ of the locally symmetric space $\Gamma \backslash \mathcal{D}$.

2) Compute the Hilbert polynomial $\mathcal{P}(k)$ of the anticanonical bundle of the compact dual $\check{\mathcal{D}}$.

3) Determine the error term $E(k)$ which is a linear polynomial in k.

We will approach these problems in this order.

Hirzebruch-Mumford volume

Gritsenko, Hulek and Sankaran gave a good description of the Hirzebruch-Mumford volume of arithmetic subgroups $\Gamma \subseteq \mathrm{O}(L)$ of an indefinite lattice L in [GHS07a]. Their result is as follows:

Proposition 10.1.4 ([GHS07a, Theorem 2.1]). *Let L be an indefinite lattice of dimension $n+2 \geq 3$. The Hirzebruch-Mumford volume of $\mathrm{O}(L)$ is*

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(L)) = \frac{2}{|g_{sp}^{+}(L)|} \cdot |\det L|^{\frac{n+3}{2}} \prod_{k=1}^{n+2} \pi^{-k/2}\,\Gamma(k/2) \prod_{p} \alpha_p(L)^{-1}$$

with g_{sp}^{+} the number of spinor genera in the genus of L, $\Gamma(z)$ the usual Gamma function and $\alpha_p(L)$ the p-adic local density of L. For arithmetic $\Gamma \subseteq \mathrm{O}(L)$ one has

$$\mathrm{Vol}_{\mathrm{HM}}(\Gamma) = [\mathrm{PO}(L) : \mathrm{P}\Gamma] \cdot \mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(L))$$

with $\mathrm{PO}(L)$ and $\mathrm{P}\Gamma$ the images of the respective groups in $\mathrm{Aut}(\mathcal{D})$.

Furthermore, the authors provide explicit formulas for the computation of the local densities for L being of signature $(2,n)$ and splitting at least one (unscaled) hyperbolic plane over the integers, cf. [GHS07a, Section 3].

Hilbert polynomial of the compact dual

The Hilbert polynomial of the compact dual $\check{\mathcal{D}}$ can be computed quite easily:

Proposition 10.1.5. *The Hilbert polynomial of the anticanonical bundle $(\omega_{\check{D}}^{n})^{-1}$ of the compact dual $\check{\mathcal{D}} \cong \mathrm{SO}(2+n)/\mathrm{SO}(2) \times \mathrm{SO}(n)$ of the symmetric space*

$$\mathcal{D}_L \cong \mathcal{D} \cong \mathrm{SO}^{+}(2,n)/\mathrm{SO}(2) \times \mathrm{SO}(n)$$

is

$$\mathcal{P}_{\check{\mathcal{D}}}(k) = \chi\left(\left(\left(\omega_{\check{\mathcal{D}}}^{n}\right)^{-1}\right)^{\otimes k}\right) = \binom{(n+1)(k+1)}{(n+1)k} - \binom{(n+1)(k+1)-2}{(n+1)k-2}.$$

Proof. We have $\mathcal{D} \cong \mathrm{SO}^{+}(2,n)/\mathrm{SO}(2) \times \mathrm{SO}(n) \hookrightarrow \mathrm{SO}(2+n)/\mathrm{SO}(2) \times \mathrm{SO}(n)$. Using the realizations

$$\mathcal{D} \cong \{[Z] \in \mathbb{P}(\mathbb{C}^{2+n}) |\, q_L(Z) = 0 \text{ and } (Z, \overline{Z}) > 0\}$$

and

$$\check{\mathcal{D}} \cong \{[Z] \in \mathbb{P}(\mathbb{C}^{2+n}) |\, q_L(Z) = 0\},$$

we see that $\check{\mathcal{D}}$ is given by a quadric (i.e. $q_L(Z) = 0$) in the projective space $\mathbb{P}(\mathbb{C}^{2+n})$. These are easily seen to be true by interpreting the first space as the Grassmannian of positive definite two-dimensional real subspaces of $\mathbb{R}^{2,n}$ and the second space as the Grassmannian of real oriented two-dimensional subspaces of $\mathbb{R}^{2+n}$.

We have the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}} \longrightarrow \mathcal{O}_{\check{\mathcal{D}}} \longrightarrow 0;$$

twisting this by $\mathcal{O}(k)$ we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(k-2) \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(k) \longrightarrow \mathcal{O}_{\check{\mathcal{D}}}(k) \longrightarrow 0$$

and using the additivity of the Euler characteristic in short exact sequences yields

$$\chi(\mathcal{O}_{\check{\mathcal{D}}}(k)) = \chi(\mathcal{O}_{\mathbb{P}^{n+1}}(k)) - \chi(\mathcal{O}_{\mathbb{P}^{n+1}}(k-2));$$

the Hilbert polynomial of the projective space is well-known to be

$$\chi(\mathcal{O}_{\mathbb{P}^n}(k)) = \binom{n+k}{k}.$$

Finally we have the following identity of line bundles

$$\left(\left(\omega^n_{\check{\mathcal{D}}}\right)^{-1}\right)^{\otimes k} = \left((\mathcal{O}_{\check{\mathcal{D}}}(-n-1))^{-1}\right)^{\otimes k} = \mathcal{O}_{\check{\mathcal{D}}}(n+1)^{\otimes k} = \mathcal{O}_{\check{\mathcal{D}}}((n+1)k)$$

and piecing everything together we get the stated result. □

This gives a preliminary dimension formula:

Lemma 10.1.6. *For $k \geq 2$ we have*

$$\begin{aligned}\dim(S_{nk}(\Gamma)) - E(k) = \frac{2[\mathrm{PO}(L) : \mathrm{P}\Gamma]}{|g^+_{sp}(L)|} |\det L|^{\frac{n+3}{2}} \prod_{k=1}^{n+2} \pi^{-k/2}\, \Gamma(k/2) \prod_p \alpha_p(L)^{-1} \\ \cdot \left(\binom{(n+1)k}{(n+1)(k-1)} - \binom{(n+1)k-2}{(n+1)(k-1)-2}\right)\end{aligned}$$

with a polynomial $E(k)$ of degree 1 and the other objects as in the preceding results.

The right side of this formula can be considered as the canonical part of the dimension formula as this does not depend on the actual choice of the toroidal compactification. Furthermore this is almost independent of the arithmetic subgroup Γ with only the index $[\mathrm{PO}(L) : \mathrm{P}\Gamma]$ to be taken into account.
In contrast, the computation of the error term $E(k)$ is sensitive to this choice and therefore more difficult and involved. Its computation will occupy us for the rest of this thesis.
In the next section we will examine the error term $E(k)$ and us the functoriality as in section 4.3 to get a first classification of the appearing terms.

10.2. Compactification-dependent computations

We showed in section 4.3 that $E(k)$ carries a lot of structure: Applying proposition 4.3.6 we see that $E(k)$ equals

$$\sum_{\substack{|\underline{b}|\geq 1\\ \text{multiplicity free}}} \lambda_{\underline{b}}\chi\left(D^{\underline{b}}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{D^{\underline{b}}}\right)$$
$$+ \sum_{\substack{|\underline{b}|\geq 1\\ \text{not}\\ \text{multiplicity free}}} \lambda_{\underline{b}}[D^{\underline{b}}]Q_{n-|\underline{b}|}\left(c_1(\Omega^1_{\overline{X}}(\log\Delta)^{\otimes(1-k)}); c_1(\Omega^1_X(\log)), \ldots, c_{n-|\underline{b}|}(\Omega^1_X(\log))\right)$$

with $\Delta = \{D_i | i \in I\}$ the collection of the irreducible boundary divisors in $\overline{X}^{\text{tor}}_{\Sigma} \setminus X$ and $\Omega^1_{\overline{X}}(\log\Delta)^{\otimes(1-k)}$ the corresponding logarithmic cotangent bundle. The multi-index notation $D^{\underline{b}}$ is as in section 4.3; the definition and some values of $\lambda_{\underline{b}}$ can be found there as well.
While the terms of the first sum carry obvious geometric meanings as Euler characteristics, the terms of the second sum need more interpretation.
We will show that there are only two types of terms to consider in the second, non-multiplicity-free case under some natural conditions: We assume from now on that $\overline{X}^{\text{tor}}_{\Sigma}$ is a smooth projective toroidal compactification such that any toroidal boundary divisor of zero-dimensional type is a smooth compact toric variety and the intersection product of the classes of distinct toroidal boundary divisor of one-dimensional type is zero. These conditions allows us to reduce the complexity of the non-multiplicity-summands even more: Any term in the second sum in the preceding decomposition of $E(k)$ is

i) either a (self-)intersection product of toroidal boundary components or

ii) a local Euler characteristic over self-intersections of toroidal boundary components of one-dimensional type.

We will define in a moment what we exactly mean by this.
The following lemma is crucial for this reduction:

Lemma 10.2.1. *Let $[D^{\underline{b}}]$ contain at least one toroidal boundary divisor D_i of zero-dimensional type with $b_i > 0$ which is a smooth compact toric variety and let $n - |\underline{b}| > 0$. Then*

$$[D^{\underline{b}}]Q_{n-|\underline{b}|}\left((1-k)c_1\left(\Omega^1_{\overline{X}}(\log)\right); c_1\left(\Omega^1_X(\log)\right), \ldots, c_{n-|\underline{b}|}\left(\Omega^1_X(\log)\right)\right) = 0.$$

Proof. We use the abbreviation

$$\mathrm{c}\left(\Omega^1_{\overline{X}}(\log\Delta)\right) = \left(c_1\left(\Omega^1_X(\log\Delta)\right), \ldots, c_{n-|\underline{b}|}\left(\Omega^1_X(\log\Delta)\right)\right)$$

and see

$$\begin{aligned}
& [D^{\underline{b}}]Q_{n-|\underline{b}|}\left((1-k)c_1\left(\Omega^1_X(\log\Delta)\right);\mathrm{c}\left(\Omega^1_X(\log\Delta)\right)\right) \\
= \ & [D^{\underline{b}'}][D_i]Q_{n-|\underline{b}|}\left((1-k)c_1\left(\Omega^1_X(\log\Delta)\right);\mathrm{c}\left(\Omega^1_X(\log\Delta)\right)\right) \\
= \ & [D^{\underline{b}'}]Q_{n-|\underline{b}|}\left((1-k)c_1\left(\Omega^1_{D_i}(\log\Delta\cap D_i)\right);\mathrm{c}\left(\Omega^1_{D_i}(\log\Delta\cap D_i)\right)\right)
\end{aligned}$$

by the result of proposition 4.3.4 about the restriction of logarithmic Chern classes to divisors and the fact that D_i is a smooth compact toric variety by assumption. The elements of $\Delta \cap D_i$ are exactly the toric boundary divisors of D_i, so the bundle

$$\Omega^1_{D_i}(\log(\Delta \cap D_i))$$

is trivial and has trivial logarithmic Chern classes by example 4.1.23. This forces the intersection product in question to be zero, as claimed. □

Using the previous result we can conclude:

Lemma 10.2.2. *If any toroidal boundary divisor of zero-dimensional type is a smooth compact toric variety and the intersection product of the classes of distinct toroidal boundary divisor of one-dimensional type is zero we have*

$$* := [D^{\underline{b}}]Q_{n-|\underline{b}|}\left((1-k)c_1\left(\Omega^1_X(\log)\right);\mathrm{ch}\left(\Omega^1_X(\log)\right)\right) = 0$$

unless

(1) $* = [D^{\underline{b}}]$ *with* $|\underline{b}| = n$, *an intersection product of irreducible toroidal boundary divisors or*

(2) $* = [D^l]Q_{n-l}\left((1-k)c_1\left(\Omega^1_X(\log)\right);\mathrm{ch}\left(\Omega^1_X(\log)\right)\right)$ *with* $n > l > 0$ *and* D *a toroidal boundary divisor of one-dimensional type.*

Proof. We'll do a case-by-case analysis of the possibilities for $\underline{b}$ in $*$:

i) $|\underline{b}| = n$: This is just case (1).

ii) $|\underline{b}| < n$: Every term contains at least one logarithmic Chern class and since we have by assumption $|\underline{b}| > 0$, the product contains at least one toroidal boundary divisor as a factor. We distinguish the different cases of types of $[D^{\underline{b}}]$ appearing here:

   a) The product $[D^{\underline{b}}]$ contains at least one $[D_i]$ of zero-dimensional type: This is the case covered by lemma 10.2.1 and hence $* = 0$.

   b) All of the appearing toroidal boundary divisors are of one-dimensional type. There are two cases:

      1) The product is pure self-intersection, i.e. $[D^{\underline{b}}] = [D^l]$ for $l = |\underline{b}|$. This corresponds to case (2).

2) There are at least two non-equal toroidal boundary divisors $D_i \neq D_j$ of one-dimensional type, which multiply to zero, so $* = 0$

This proves the claim. $\square$

Terms as in case (2) are called *local Euler characteristics* since we will describe them later on as being essentially Euler characteristics on (models of) D^l.
Note that we do not claim that terms of type (1) or (2) differ from zero (many of them will not); this criterion is just sufficient.

Remark 10.2.3. All in all, this reduces the computation of $E(k)$ to compute the following

i) the Euler characteristic $\chi\left(D^{\underline{b}}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{D^{\underline{b}}}\right)$ for $[D^{\underline{b}}]$ multiplicity-free and containing at most one toroidal boundary divisor of one-dimensional type

ii) the pure self-intersection product $[D^{\underline{b}}]$ with $|\underline{b}| = n$ satisfying $b_i \geq 2$ for all i and such that $\lambda_{\underline{b}} \neq 0$ (note that non-multiplicity-free pure intersection products $D^{\underline{b}}$ with $b_i = 1$ have $\lambda_{\underline{b}} = 0$)

iii) the *local Euler characteristic* $[D^l]Q_{n-l}\left((1-k)c_1\left(\Omega^1_{\overline{X}}(\log)\right); \mathrm{ch}\left(\Omega^1_{\overline{X}}(\log)\right)\right)$ for D a toroidal boundary divisor of one-dimensional type and $0 < l < n$.

We know that $E(k)$ is a linear polynomial in k and it is possible to decompose it via the degree of k. Obviously, the pure intersection products do not depend on k and hence contribute to the constant coefficient of $E(k)$. The terms in i) of remark 10.2.3 only seemingly contribute to k:

Proposition 10.2.4. *If $[D^{\underline{b}}]$ is multiplicity-free, we have*

$$\chi\left(D^{\underline{b}}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{D^{\underline{b}}}\right) = 1.$$

Proof. This is again an application of example 4.1.23: The Hirzebruch-Riemann-Roch theorem (cf. theorem 4.1.15) shows that the Euler characteristic of the sheaf in question depends only on its Chern classes; since we are dealing with a line bundle, all but the first one

$$c_1\left(\Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{D^{\underline{b}}}\right) = (1-k)c_1\left(i^*\left(\Omega^1_{\overline{X}}(\log\Delta)\right)\right)$$

of these vanish. With repeated use of proposition 4.3.4 we see

$$c_1\left(i^*\left(\Omega^1_{\overline{X}}(\log\Delta)\right)\right) = c_1\left(\Omega^1_{D^{\underline{b}}}(\log\Delta')\right)$$

for

$$\Delta' = \{D \cap (D_1 \cap \ldots \cap D_l) \,|\, D \in \Delta \setminus \{D_1, \ldots, D_l\}\}.$$

The collection Δ' consists exactly of the irreducible torus-invariant divisors on the smooth compact toric variety $D^{\underline{b}}$ (compare lemma 9.2.3), so example 4.1.23 shows that the bundle $\Omega^1_{D^{\underline{b}}}(\log\Delta')$ is trivial and its first Chern class vanishes; obviously, this shows

$$c_1\left(\Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{D^{\underline{b}}}\right) = 0$$

and its Euler characteristic is

$$\chi\left(D^{\underline{b}}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{D^{\underline{b}}}\right) = \chi\left(D^{\underline{b}}, \mathcal{O}_{D^{\underline{b}}}\right) = 1$$

by proposition 2.1.22. □

In total we see that the only terms contributing to the linear coefficient of $E(k)$ are of the form $[D^l]Q_{n-l}(\dots)$ for $l > 0$ and D a toroidal boundary divisor of one-dimensional type as in (ii) of remark 10.2.3. Hence:

Proposition 10.2.5. *We have*

$$E(k) = \sum_{l=1}^{n-1} \lambda_{(l)} \sum_{\substack{D \text{ of one-} \\ \text{dimensional type}}} [D^l]Q_{n-l}\left(c_1\left(\Omega^1_{\overline{X}}(\log\Delta)^{\otimes(1-k)}\right); \mathrm{c}\left(\Omega^1_X(\log)\right)\right) + \mathcal{O}(1).$$

This ends our treatment of general orthogonal locally symmetric spaces corresponding to even lattice of signature $(2, n)$. In the next and final part of this thesis we will treat the special case of the 12-dimensional lattices $II_{2,10}(N)$ that we used extensively as an example in earlier chapters.

Part III.

The reflective compactification of $\mathbb{I}_{2,10}(\mathrm{N})$

11. Construction of the reflective compactification

In this third and final part of this thesis we will apply the theory developed in the preceding chapters to a particularly nice class of lattices and their corresponding orthogonal locally symmetric spaces: the rescalings

$$II_{2,10}(N) = II_{1,1}(N) \oplus II_{1,1}(N) \oplus \mathrm{E}_8(-N)$$

of the unique even unimodular lattice $II_{2,10}$ for $N \geq 1$ which already served as an example several times before.
In this chapter we will apply the theory of toroidal compactifications to construct and describe an especially well-behaved toroidal compactification of these locally symmetric spaces.
This chapter is structured as follows: The first section gives a rough overview of the theory of Coxeter groups and hyperbolic reflection groups; in the second section we construct the aforementioned very special toroidal compactification with the help of this theory. The third section is concerned with the closures of certain Heegner divisors in this toroidal compactification.

11.1. Coxeter theory

We follow the treatment in [AB08] where the results of this section are taken from.
We saw that toroidal compactifications in the case of an orthogonal symmetric space associated to a lattice of signature $(2, n)$ depend only on $\overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$-admissible cone decompositions of the rational closure of the cones $\mathcal{C}(\mathcal{F})$ for zero-dimensional rational boundary components $\mathcal{F}$ of $\mathcal{D}_L$. Each one of these lives inside the Lorentzian space $\mathcal{U}(\mathcal{F}) \cong \mathbb{R}^{1,n-1}$, where $\overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$ acts as a subgroup of the automorphism group of $\mathcal{C}(\mathcal{F})$. In particular: The group $\overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$ consists of (reflection) symmetries of the Lorentzian space, so it is worthwhile to get a better understanding and theoretical background of Lorentzian spaces and their automorphism groups.
We embed this into the context of the more general *Coxeter theory* which deals with certain abstract groups given in terms of generators and relations. These groups are a natural generalization of reflection groups and are therefore useful for our task.
The central objects of Coxeter theory are the eponymous *Coxeter groups* which were invented as a formal characterization of groups generated by reflections.

Definition 11.1.1. Let W be a finitely generated group. The group W is called a *Coxeter group* if there is a finite generating set $S = \{s_1, \ldots, s_n\} \subseteq W$ of elements of order 2 such that W admits a presentation of the form

$$W = \left\langle S \,|\, (s_i s_j)^{m(s_i,s_j)} = 1 \right\rangle$$

where $m(s_i, s_j)$ is the order of $s_i s_j$ in W and there is a relation for any pair s_i, s_j with $m(s_i, s_j) < \infty$. The pair (W, S) is called a *Coxeter system*.

There is a very convenient way of storing the data in a Coxeter system:
The relations of the generators of a Coxeter group can be encoded by the *Coxeter matrix* $M = M_{W,S}$ with

$$(M_{W,S})_{ij} = (m(s_i, s_j))_{ij}$$

or the *Schläfli matrix* C with

$$C_{ij} = -2\cos\left(\frac{\pi}{M_{ij}}\right).$$

One can show that, up to isomorphism, W is determined by its Coxeter or Schläfli matrix (cf. [AB08, Corollary 2.35]). Obviously, the corresponding matrices are symmetric since $m(s_i, s_j) = m(s_j, s_i)$. The diagonal of a Coxeter matrix consists of 1's while the diagonal of a Schläfli matrix has every entry equal to $-2\cos(\pi) = 2$.
Note that this construction attaches a symmetric Coxeter matrix with unit diagonal and non-diagonal entries in $\{2, 3, \ldots\}$ to any Coxeter system. As we will see in a moment, the converse of this is also true and yields an equivalence of concepts.
Another even more useful graphical description of a Coxeter group is by a *Coxeter-Dynkin diagram* which is a graph that can be constructed by the following rules:

(i) Draw one vertex for each generator and label it with its subscript.

(ii) Connect two vertices with an edge if and only if $m_{ij} \geq 3$.

(iii) Label the edges by m_{ij} if $m_{ij} \geq 4$.

To illustrate this, we consider the symmetry group of a regular n-gon:

Example 11.1.2. The n-th dihedral group D_n of order $2n$ is given by the presentation

$$D_n = \left\langle \{s_1, s_2\} \,|\, s_1^2 = 1, s_2^2 = 1, (s_1 s_2)^n = 1 \right\rangle$$

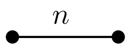
with set of generators $S = \{s_1, s_2\}$. The Coxeter matrix is

$$M = \begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}$$

and the *Schläfli matrix* is

$$C = \begin{pmatrix} 2 & -2\cos(\pi/n) \\ -2\cos(\pi/n) & 2 \end{pmatrix}.$$

The corresponding Dynkin diagram is



As noted in the introduction: Coxeter groups are constructed as the formalization of reflection group, so it is no surprise that any sensible reflection group of a geometric object turns out to be a Coxeter group; moreover, the reverse is also true! Indeed, there is, for any Coxeter group W, a canonical representation via which W acts as a reflection group on some vector space. This representation can be used to show that there is indeed an equivalence of *abstract Coxeter matrices* (i.e. symmetric matrices M with $M_{ii} = 1$ and $M_{ij} \in \mathbb{Z}_{>1} \cup \{\infty\}$) by the procedure as above: At the moment it is only clear how to construct the Coxeter matrix of a Coxeter group, but not that, given an abstract Coxeter matrix M, the group defined by the presentation induced by M is a Coxeter group W_M with Coxeter matrix M.
We will construct this canonical linear representation in the following:

Lemma 11.1.3. *Let (W, S) be a Coxeter system. Define a vector space $V = \mathbb{R}^S$ with standard basis $(e_s)_{s\in S}$ and equip it with a bilinear symmetric form B via*

$$B(e_{s_1}, e_{s_2}) = -\cos\left(\frac{\pi}{m\left(s_1, s_2\right)}\right).$$

There is an injective homomorphism $\rho : W \to \mathrm{GL}(V)$ induced by

$$\rho(s_i) = \left(v \mapsto \sigma_{s_i}(v) = s_i.v = v - \frac{2B(e_{s_i})}{B(e_{s_i}, e_{s_i})} e_{s_i}\right)$$

for $s_i \in S$.

The faithful representation of lemma 11.1.3 is the *canonical linear representation* of W. Furthermore, we can define a dual representation of W on V^*, the dual space of V by

$$w.v^* = \left(u \mapsto B(v, w^{-1}.u)\right).$$

In this case, a generator $s \in W$ acts on V^* as the linear reflection with fixed hyperplane $H_s = B(\cdot, e_s) = 0$. The entries of the Schläfli can then be thought of as the angles between the hyperplanes. We even get:

Lemma 11.1.4 ([AB08, Corollary 2.68])**.** *The Coxeter group W with given set of generators S is finite if and only if its Schläfli matrix is positive definite.*

The set-theoretic complement of the hyperplanes H_s for $s \in S$ decompose

$$V^* \setminus \bigcup_{s\in S} H_s$$

into disconnected components and we choose one as follows:

Definition 11.1.5. The *fundamental chamber* of V^* is the cone C defined by the intersection of the half-spaces $B(\cdot, e_s) > 0$ for $s \in S$.

This is a strongly convex polyhedral cone in the sense of definition 2.1.3.
Note that the bases of V and V^* are far from unique, so this is only a distinguished choice with respect to a given basis. On the contrary, any choice of an intersection of half-spaces $\pm B(\cdot, e_s) > 0$ is a fundamental chamber for a suitable choice of basis: Simply choose $\pm e_s$ as part of the basis if the intersection is defined by $\pm B(\cdot, e_s) > 0$. Even more general: Any W-image of the distinguished basis $(e_s)_{s\in S}$ yields a basis of $\mathbb{R}^s$ and with it a fundamental chamber C.
We will adopt the following abuse of notation:
Since we are given a distinguished non-degenerate bilinear form on V, we will identify V and V^* via this form. This is in accordance with our abuse of notations for lattices and their duals in chapter 5 where we did the same.
To deal with the non-uniqueness of the fundamental chamber we provide the following terminology for the other possible generators and fundamental chambers:

Definition 11.1.6. The vectors $w.e_s$ for $w \in W$ and $s \in S$ are called *roots*, the hyperplanes $H_{ws} = w.H_s \subseteq V$ are *walls* and the $w.C$ are *chambers*. Any wall H_{ws} defines two half-spaces $H_{ws}^{\pm}$ of V by $\pm B(\cdot, w^{-1}e_s) > 0$.
The *Tits cone* is

$$X = \bigcup_{w\in W} w.\overline{C}$$

or, equivalently, the union of all W-images of all the faces of C; these images are called *cells* of X.

Note that this differs from the literature: [AB08] defines a cell to be the relative interior of the cells we defined here.
This cone is convex and any chamber is a strict fundamental domain for the action of W on X. We can say even more about it:

Proposition 11.1.7 ([AB08, Theorem 2.80])**.** *The Tits cone X is convex. Moreover:*

(i) For any $x, y \in \overline{C}$ with $w.x = y$ for some $w \in W$ one finds $x = y$ and

$$w \in \mathrm{Stab}_x = \langle s \in S \mid s.x = x \rangle .$$

(ii) For any pair (A, H) consisting of the relative interior A of a cell and a wall H, one has either $A \subseteq H$ or $A \subseteq H^+$ or $A \subseteq H^-$.

In particular, the Tits cone is strongly convex and the intersection of a cell with a wall is a face of the cell.

This can be used to show that the relative interiors of any two cells are disjoint. An easy consequence is that the intersection of two cells is given by the union of their common faces.

Another useful concept for using induction on Coxeter groups is the notion of *length* of an element. This allows to speak about a distance of a chamber to a given fundamental chamber and lays ground for the use of induction on the Tits cone.

Definition 11.1.8. Let (W, S) be a Coxeter group with S a set of generators and let $w \in W$. A decomposition of w is any finite sequence $(s_1, s_2, \dots, s_k)$ in S such that $w = s_1 s_2 \dots s_k$. The *length* $l(w) \in \mathbb{N}_{\geq 0}$ is the minimal length k of decomposition of w. The length of $1 \in W$ is $L(1) = 0$.

Obviously $l(s_i) = 1$ for any $s \in S$. Note that cancellation can occur, so that in general $l(w_1 w_2) \neq l(w_1) + l(w_2)$, e.g. for $w_1 = w_2 = s \in S$ we have

$$l(s^2) = l(1) = 0 < 2 = l(s) + l(s).$$

The next lemma shows that the action of an element on the length of another element is connected with the geometry of the Coxeter group:

Lemma 11.1.9 ([AB08, Lemma 2.58])**.** *For fixed $s \in S$ and arbitrary $w \in W$ we have*

$$wC \subseteq \begin{cases} H_s^+ & \text{if } l(sw) = l(w) + 1 \\ H_s^- & \text{if } l(sw) = l(w) - 1 \end{cases}.$$

This covers the necessary ground for a treatment of hyperbolic reflection groups which are examples of Coxeter groups.

Hyperbolic reflection groups

This section follows closely the treatment in [Hec18]. Some parts of the theory and the appearing notions are running parallel to the one just introduced. Towards the end of this short exposition we will show the compatibility of these concepts and hence the deeper reason for giving some of the objects the same names.
Due to the obvious isomorphism $q \mapsto -q$ we assume any Lorentzian lattice (L, q) to be of signature $(1, n)$ with $n \geq 1$.

Definition 11.1.10. A Lorentzian lattice L with bilinear form $(\cdot, \cdot)$ is called a *root lattice* if it is spanned by its *roots*, that is, by the elements

$$R(L) = \left\{ \alpha \in L \,\middle|\, \alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} \in L^* \right\}.$$

For an even lattice, the roots are exactly the vectors of norm -2. The most important example for our purposes is the following special lattice:

Example 11.1.11 ([Con83])**.** We denote by $\mathrm{E}_{10}(-1)$ the (-1)-scaling of the unique unimodular Lorentzian lattice in the genus $II_{1,9}$ and realize it as a lattice in $\mathbb{R}^{1,9}$ with quadratic form

$$-(x_1^2 + \dots + x_9^2) + x_{10}^2$$

as follows: It is given by

$$\mathrm{E}_{10}(-10) = \left\{ x = (x_1, \ldots, x_{10}) \in \mathbb{Z}^{10} \cup \left(\frac{1}{2} + \mathbb{Z}\right)^{10} \mid \sum_{i=1}^{10} x_i \in 2\mathbb{Z} \right\} \subseteq \mathbb{R}^{1,9}$$

and spanned by the vectors $\lambda \in \mathrm{E}_{10}(-1)$ of norm $q(\lambda) = -2$ with

$$(\lambda, (0, 1, \ldots, 8, 38)) = 1,$$

so it is a root lattice.

Note that we will use the notations $\mathrm{E}_{10}(-1)$ and $I\!I_{1,9}$ interchangeably since they denote the same objects. The former is usually more common in the analytical theory with fixed system of coordinates and in more physical interpretations while the latter is mainly used in the theory of lattices.
From now on, let L be a unimodular Lorentzian root lattice of signature $(1, n)$. Any root $\alpha \in L$ defines an automorphism of L by the reflection

$$\sigma_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha = \lambda - (\lambda, \alpha)\alpha = \lambda - k\alpha \in L$$

for some $k \in \mathbb{Z}$ by the integrality of L.
A root $\alpha \in R(L)$ defines also an automorphism on the real vector space $L \otimes \mathbb{R}$. The group $W(L)$ generated by σ_α for $\alpha \in R(L)$ is the *reflection group* of L. It is a discrete subgroup of $\mathrm{O}(V)$. The lattice L is called *reflective* if $W(L)$ has finite index in $\mathrm{O}(L)$.
Moreover, $W(L)$ acts also on hyperbolic n-space H^n by the following construction: The *hyperboloid model* of H^n is given by one connected component of the *time-like cone*

$$\{v \in V \mid (v, v) = +1\}$$

and the action of σ_α on V induces an action on H^n. Consider the *forward time-like cone*

$$V_+ = \mathbb{R}_+ H^n.$$

Every σ_α has its associated *mirror* or *wall*

$$H_{\sigma_\alpha} = H_\alpha = \{v \in V \mid (v, \alpha) = 0\},$$

a hyperplane in V which induces a hyperplane in H^n be intersection.
A vector $v \in V_+ \setminus \bigcup_{\alpha \in R(L)} H_\alpha$ is called *regular* and the set of all regular vectors is denoted by V_+°. Note that $\bigcup_{\alpha \in R(L)} H_\alpha$ is locally finite by the discreteness of $W(L)$, so regular elements are dense in V.
We choose one of the connected components (called *chambers*) of the set of regular elements and denote it by C_+; this will be called the *fundamental Weyl chamber*; it is a convex polyhedral cone. It defines the set of *positive roots* with respect to that fundamental chamber as

$$R_+(L) = \{\alpha \in R(L) \mid (\alpha, C_+) > 0\}.$$

A positive root is *simple*, if it cannot be written as $\lambda_1\alpha_1 + \lambda_2\alpha_2$ with $\alpha_1, \alpha_2 \in R_+(L)$ and $\lambda_1, \lambda_2 \geq 1$.
A set of simple roots can be determined by *Vinberg's algorithm* (cf. [Vin75]). For the case of the E_{10}-lattice we can state the following:

Example 11.1.12. A system of simple roots for $\mathrm{E}_{10}(-1)$ is given by the vectors

$$\begin{array}{lcllllllllllll}
\rho_1 & = & (& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 &) \\
\rho_2 & = & (& 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 &) \\
\rho_3 & = & (& 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 &) \\
\rho_4 & = & (& 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 &) \\
\rho_5 & = & (& 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 &) \\
\rho_6 & = & (& 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 &) \\
\rho_7 & = & (& 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &) \\
\rho_8 & = & (& -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &) \\
\rho_9 & = & (& 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 &) \\
\rho_{10} & = & (& 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &)
\end{array}$$

We will call these roots *fundamental roots* of $\mathrm{E}_{10}(-1)$. It is worth noting that the fundamental roots constitute a basis of $\mathrm{E}_{10}(-1)$, which again shows that $\mathrm{E}_{10}(-1)$ is a root lattice. Additionally, the lattice $\mathrm{E}_{10}(-1)$ is reflective.
The Gram matrix corresponding to this basis is

$$A = -\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix}.$$

Moreover, we consider its dual basis $(w_1, \dots, w_{10})$ with $(\rho_i, w_j) = \delta_{ij}$, whose elements are the *fundamental weights* of $\mathrm{E}_{10}(-1)$. Their negatives are given as

$$\begin{array}{lcllllllllllll}
w_1 & = & (& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 &) \\
w_2 & = & (& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 &) \\
w_3 & = & (& 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 &) \\
w_4 & = & (& 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 4 &) \\
w_5 & = & (& 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 5 &) \\
w_6 & = & (& 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 6 &) \\
w_7 & = & (& 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 &) \\
w_8 & = & (& 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 9/2 &) \\
w_9 & = & (& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 &) \\
w_{10} & = & (& 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 7/2 &)
\end{array}$$

with the Gram matrix

$$A^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 & 3 \\ 1 & 2 & 4 & 6 & 8 & 10 & 12 & 8 & 4 & 6 \\ 2 & 4 & 6 & 9 & 12 & 15 & 18 & 12 & 6 & 9 \\ 3 & 6 & 9 & 12 & 16 & 20 & 24 & 16 & 8 & 12 \\ 4 & 8 & 12 & 16 & 20 & 25 & 30 & 20 & 10 & 15 \\ 5 & 10 & 15 & 20 & 25 & 3 & 36 & 24 & 12 & 18 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 28 & 14 & 21 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 18 & 9 & 14 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 9 & 4 & 7 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 14 & 7 & 10 \end{pmatrix}.$$

Back to the general case of an unimodular Lorentzian lattice L we can state the following properties of the reflection group $W(L)$, cf. [Hec18, Section 5.5]:

Proposition 11.1.13. *The walls of C_+ are given by*

$$\overline{C_+} \cap H_\alpha \text{ for } \alpha \in R_+(L)$$

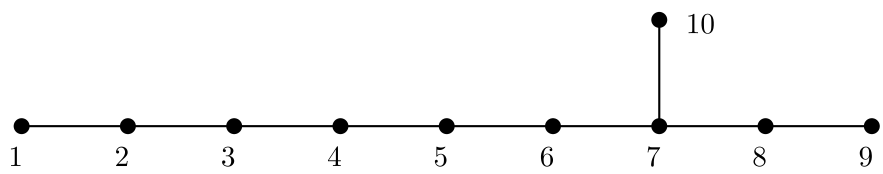
and the reflection group $W = W(L)$ is generated by the reflections in the simple roots $\rho_i \in R_+(L)$. It is isomorphic to the Coxeter group corresponding to the Schläfli matrix G_W given by

$$(G_W)_{i,j} = ((\rho_i, \rho_j))_{i,j}.$$

The cone C_+ is spanned by the fundamental weights w_i; these are the primitive lattice generators of the one-dimensional faces of C_+ defined by the intersection of all but one walls of C_+ (the distinguished missing one is the one for the reflection in ρ_i).

Applying this to our example $\mathrm{E}_{10}(-1)$ gives the following explicit description of the fundamental Weyl chamber and the reflection group $W(\mathrm{E}_{10}(-1))$.

Example 11.1.14. The fundamental Weyl chamber for $\mathrm{E}_{10}(-1)$ has 10 walls and is spanned by the 10 fundamental weights $w_1, \ldots, w_{10}$ of example 11.1.12. The Coxeter-Dynkin diagram corresponding to the reflection group $W(\mathrm{E}_{10}(-1))$ of $\mathrm{E}_{10}(-1)$ is:



We are now able to bring the worlds of Coxeter groups and Lorentzian lattices together and resolve the parallel construction of concepts:

Proposition 11.1.15. *Let L be a unimodular Lorentzian lattice with a choice $\rho_1, \ldots, \rho_k$ of fundamental roots and corresponding fundamental Weyl chamber.*

The Tits cone Y of the reflection group $W(L)$ can be identified with

$$\overline{V}_{+}^{rat} = V_{+} \sqcup \mathbb{R}_{+}\partial V_{+} \sqcup \{0\},$$

the rational closure *of V_{+}. Here ∂V_{+} is the union of the isotropic sublattices of L. In particular: The closure $\overline{C_0}$ of the fundamental Weyl chamber C_0 is a strict fundamental domain for the action of $W(L)$ on Y and the stabilizer of a point is generated by the reflections in the walls containing it. Moreover, for any $W(L)$-image A of a face of C_0 and any $W(L)$-image H of a wall of C_0, A is either contained in H or in one the corresponding half-spaces. All in all, the notions of roots, weights, walls chamber, etc. for a Lorentzian lattice L and its reflection group $W(L)$ (considered as a Coxeter group with generating set $\rho_1 \ldots, \rho_k$) can be identified.*

Proof. Identify the simple roots ρ_i of L with the basis e_{s_i} of the vector space V of the canonical linear representation and keep in mind the identifications of L and L' as well as of V and V^*, then this is mostly by simple comparison of the constructions in the preceding sections. The statement about the rational closure follows from [Hec18, Exercise 4.52]. □

11.2. The reflective compactification of $\mathrm{X}(\mathbf{I\!I}_{2,10}(\mathrm{N}))$

We construct a natural toroidal compactification which we will call the *reflective compactification* due to its origin in the theory of reflection groups just presented.
We will focus from now on on the rescalings of the lattice $I\!I_{2,10}$ and their corresponding locally symmetric space $X(I\!I_{2,10}(N))$ (or, equivalently, the $\Gamma\backslash\mathcal{D}_{I\!I_{2,10}}$ with Γ a principal congruence subgroup) even if the construction may be generalized to more lattices. We start our considerations with the case of $N = 1$.

The case of $\mathbf{I\!I}_{2,10}$

Let $L = I\!I_{1,1} \oplus I\!I_{1,1} \oplus \mathrm{E}_8(-1) = I\!I_{2,10}$ be the even unimodular lattice of signature $(2, 10)$ and $\Gamma = \mathrm{SO}^{+}(I\!I_{2,10})$ its discriminant kernel. We note that there is only one Γ-orbit of zero-dimensional rational boundary components, so a symmetric toroidal compactification can be obtained by specifying a $\overline{\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}}$-admissible decomposition of $\overline{\mathcal{C}(\mathcal{F}_0)}^{\mathrm{rat}}$ for a single $\mathcal{F}_0$ by section 8.2.

Construction 11.2.1. Let $\mathcal{F}_0$ be a zero-dimensional boundary component of $\mathcal{D}$ and choose appropriate coordinates, so

$$I\!I_{2,10} = I\!I_{1,1} \oplus \mathrm{E}_{10}(-1) = I\!I_{1,1} \oplus I\!I_{1,9}.$$

The realization as a Siegel domain of the third kind with respect to $\mathcal{F}_0$ is

$$\mathcal{D}_L \cong \{x \in \mathbb{C}^n \mid \operatorname{Im} x \in \mathcal{C}(\mathcal{F}_0)\}$$

and the cone $\mathcal{C}(\mathcal{F}_0)$ is given by

$$\mathcal{C}(\mathcal{F}_0) = \{c \in II_{1,9}| \ q_{II_{1,9}}(\operatorname{Im} x) > 0, \operatorname{Im} x_1 > 0\},$$

the time-like cone with positive time-like coordinate as in section 6.3. Its rational closure $\overline{\mathcal{C}(\mathcal{F}_0)}^{\mathrm{rat}}$ is its union with the rational isotropic rays in its boundary; these correspond to the one-dimensional rational boundary components adjoint to $\mathcal{F}_0$.
The group $G_l(\mathcal{F}_0)$ is just the group of autochronous symmetries of the cone, which is nothing but

$$\operatorname{Aut}(\mathcal{C}(\mathcal{F}_0)) = \mathrm{O}^+(\mathbb{R}^{1,9}) = \mathrm{O}^+(\mathrm{E}_{10}(-1) \otimes \mathbb{R})$$

and its intersection with Γ is

$$\overline{\mathcal{P}(\mathcal{F}_0)}_{\mathbb{Z}} = \mathrm{SO}^+(\mathrm{E}_{10}(-1)),$$

so any $\mathrm{SO}^+(\mathrm{E}_{10}(-1))$-invariant decomposition of $\overline{\mathcal{C}(\mathcal{F}_0)}^{\mathrm{rat}}$ gives an Γ-admissible family. By proposition 11.1.15 the cone $\overline{\mathcal{C}(\mathcal{F}_0)}^{\mathrm{rat}}$ is the Tits cone of the hyperbolic reflection group

$$W(\mathrm{E}_{10}(-1)) = \mathrm{O}^+(\mathrm{E}_{10}(-1))$$

and the closure $\overline{C_0}$ of any fundamental Weyl chamber C_0 is a strict fundamental domain for the action of $\mathrm{O}^+(II_{1,9})$ on $\overline{\mathcal{C}(\mathcal{F}_0)}^{\mathrm{rat}}$. This fundamental Weyl chamber is a strongly convex polyhedral cone generated by its fundamental weights $w_i \in {\mathrm{E}_{10}}^{\vee}(-1) = \mathrm{E}_{10}(-1)$. Since the number of fundamental weights is finite (=10), the fundamental Weyl chamber has only finitely many faces

$$\mathbf{F} = \left\{\{0\}, \mathbb{R}_{>0}w_1, \mathbb{R}_{>0}w_2, \ldots, \overline{C_0}\right\}.$$

We define

$$\Sigma(\mathcal{F}_0) = \bigcup_{\gamma \in W(II_{1,9})} \bigcup_{\sigma \in \mathbf{F}} \{\gamma\sigma\}.$$

Note that this decomposition does not depend on the choice of the fundamental Weyl chamber, even though it appears in its construction. Nevertheless, whenever we work with $\Sigma(\mathcal{F}_0)$ constructed in this way, we assume that we have fixed a fundamental Weyl chamber C_0.
We will show in the following that $\Sigma(\mathcal{F}_0)$ is a $\overline{\mathcal{P}(\mathcal{F}_0)}_{\mathbb{Z}}$-admissible decomposition of $\overline{\mathcal{C}(\mathcal{F}_0)}^{\mathrm{rat}}$:

Lemma 11.2.2. *The collection $\Sigma(\mathcal{F}_0)$ defined as in construction 11.2.1 is a rational polyhedral partial decomposition of $\overline{\mathcal{C}(\mathcal{F}_0)}^{rat}$.*

Proof. We check the conditions in definition 2.1.7:

- The construction of C_0 shows that it is a rational strongly convex cone and faces of C_0 still have these properties; linear lattice automorphisms preserve these as well, so $\Sigma(\mathcal{F}_0)$ consists of strongly convex rational polyhedral cones.

- Let $\sigma \preceq \sigma' \in \Sigma(\mathcal{F}_0)$, then

$$\sigma = H_1 \cap \ldots \cap H_k \cap \sigma'$$

for some hyperplanes $H_i \subseteq \mathcal{U}(\mathcal{F}_0)$. By construction there is $\gamma \in \mathrm{O}^+(II_{1,9})$ with $\sigma' = \gamma\sigma''$ where $\sigma'' \preceq \overline{C_0}$ and we can see

$$\sigma = H_1 \cap \ldots \cap H_k \cap \gamma\sigma'' = \gamma(\gamma^{-1}H_1 \cap \ldots \cap \gamma^{-1}H_k \cap \sigma'')$$

where the interior of the parentheses is a face of $\sigma'' \preceq \overline{C_0}$, hence itself a face of $\overline{C_0}$, so σ is a $\mathrm{O}^+(II_{1,9})$-translate of $\sigma'' \preceq \overline{C_0}$ and therefore contained in $\Sigma(\mathcal{F}_0)$.

- Let $\sigma, \sigma' \in \Sigma(\mathcal{F}_0)$. We can assume that the intersection is non-trivial and of the form

$$\sigma \cap \sigma' = \tau \cap \gamma\tau'$$

for some $\gamma \in \mathrm{O}^+(II_{1,9})$ and $\tau, \tau' \preceq \overline{C_0}$. We can further reduce this to show that

$$\overline{C_0} \cap \gamma\overline{C_0} \preceq \overline{C_0} :$$

If this is the case, then

$$\overline{C_0} \cap \gamma\overline{C_0} = H_1 \cap \ldots H_k \cap \overline{C_0}$$

for some hyperplanes, so

$$\begin{aligned} \tau \cap \gamma\tau' &= \tau \cap \gamma H_1' \cap \ldots \cap \ldots \gamma H_k' \cap \gamma\overline{C_0} \\ &= \tau \cap \gamma H_1' \cap \ldots \cap \ldots \gamma H_k' \cap \gamma\overline{C_0} \cap \overline{C_0} \\ &= \tau \cap \gamma H_1' \cap \ldots \cap \ldots \gamma H_k' \cap H_1 \cap \ldots H_k \end{aligned}$$

which is a face of τ.

We claim

$$\overline{C_0} \cap \gamma\overline{C_0} = \overline{C_0} \cap (H_1 \cap \ldots H_k)$$

where the H_i's are the hyperplanes defining $\overline{C_0}$ (corresponding to the fundamental roots), numbered in a way such that exactly the first k of those contain $\overline{C_0} \cap \gamma\overline{C_0}$. The first inclusion is obvious. For the other inclusion we proceed as follows: Any element of $\overline{C_0} \cap (H_1 \cap \ldots H_k)$ is in $\overline{C_0} \cap \alpha\overline{C_0}$ for $\alpha \in \langle s_1, \ldots s_k\rangle$, the group generated by the reflections s_i in the walls H_i. It remains to prove $\gamma \in \langle s_1, \ldots, s_k\rangle$.

By construction, the cone $\overline{C_0}$ is a strict fundamental domain for the action of $\mathrm{O}^+(II_{1,9})$ and hence we see that γ acts trivial on $\overline{C_0} \cap \gamma\overline{C_0}$, i.e.

$$\gamma \in Z(\overline{C_0} \cap \gamma\overline{C_0}) = \bigcap_{x \in \overline{C_0} \cap \gamma\overline{C_0}} \mathrm{Stab}(x)$$

and we only need to find an $x_0 \in \overline{C_0} \cap \gamma\overline{C_0}$ with $\mathrm{Stab}(x_0) = \langle s_1, \ldots, s_k\rangle$. This is easy: Any $x \in \overline{C_0} \cap \gamma\overline{C_0}$ is in $H_1 \cap \ldots \cap H_k$ and gets stabilized by $\langle s_1, \ldots, s_k\rangle$. Analogous considerations show $(\tau \cap \gamma\tau') \preceq \gamma\tau'$.

- Almost by definition we have

$$\overline{\mathcal{C}(\mathcal{F}_0)}^{\mathrm{rat}} = \bigcup_{\gamma\in W(II_{1,9})} \gamma\overline{D_0} = \bigcup_{\gamma\in W(II_{1,9})} \bigcup_{\sigma\in\mathbf{F}} \gamma\sigma = \bigcup_{\sigma\in\Sigma(\mathcal{F}_0)} \sigma,$$

  so we get indeed a decomposition of $\overline{\mathcal{C}(\mathcal{F}_0)}^{\mathrm{rat}}$.

$\square$

We continue by proving that this decomposition is also $\overline{\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}}$-admissible:

Lemma 11.2.3. *The rational polyhedral cone decomposition* $\Sigma(\mathcal{F}_0)$ *is in fact a* $\overline{\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}}$*-admissible cone decomposition of* $\overline{\mathcal{C}(\mathcal{F}_0)}^{rat}$.

Proof. Since $\overline{\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}} = \mathrm{SO}^+(II_{1,9}) \subseteq \mathrm{O}^+(II_{1,9})$, the decomposition is closed under its action by construction. Moreover: Every cone in $\Sigma(\mathcal{F}_0)$ is $\mathrm{O}^+(II_{1,9})$-equivalent to some face of $\overline{C_0}$, which, as a polyhedral cone, has only finitely many faces. Since the index of $\mathrm{SO}^+(II_{1,9})$ in $\mathrm{O}^+(II_{1,9})$ is 2, hence finite, there are only finitely many orbits under the action of $\overline{\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}} = \mathrm{SO}^+(II_{1,9})$ on $\Sigma(\mathcal{F}_0)$. $\square$

In total, construction 11.2.1 gives an admissible fan and a corresponding toroidal compactification:

Definition 11.2.4. The symmetric toroidal compactification corresponding to the admissible family as in construction 11.2.1 is called the *reflective toroidal compactification* and Σ is the *Coxeter family*.

These objects will be central for our later work on dimension formulas of automorphic forms on $X(II_{2,10}(N))$.

The case of congruence subgroups

For $L = II_{2,10}$ and $N \geq 1$ we now consider the rescaled lattices $L(N)$. These are still of signature $(2, n)$, so an $\Gamma = \widetilde{\mathrm{SO}}^+(II_{2,10}(N))$-admissible family can be induced by a system of admissible decompositions of $\overline{\mathcal{C}(\mathcal{F})}^{\mathrm{rat}}$ for $\mathcal{F}$ running through a system of Γ-representatives of rational zero-dimensional boundary components $\mathcal{F}$ of $\mathcal{D}_{L(N)}$ as in construction 8.2.1. Even better: The supergroup

$$\mathrm{O}^+(II_{2,10}(N)) = \mathrm{O}^+(II_{2,10}(N)) \supseteq \widetilde{\mathrm{SO}}^+(II_{2,10}(N))$$

acts with a single orbit on these boundary components, so we can induce a Γ-admissible family from $\overline{\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}}$-admissible decomposition of $\overline{\mathcal{C}(\mathcal{F}_0)}^{\mathrm{rat}}$ for a single zero-dimensional boundary component $\mathcal{F}_0$.
We want to use the decomposition from construction 11.2.1 for this:
As before we can identify $\mathcal{D}_L \cong \mathcal{D}_{L(N)}$ and get identifications of the rational boundary components and of all real structures corresponding to them; the rational structure differs

by a factor of $\sqrt{N}$. In particular: The cone $\overline{\mathcal{C}(\mathcal{F}_0)}^{\text{rat}}$ for $\mathcal{F}_0$ a zero-dimensional boundary component of $\mathcal{D}_{L(N)}$ is exactly the same cone as $\overline{\mathcal{C}(\mathcal{F}_0')}^{\text{rat}}$ for $\mathcal{F}_0'$ for the corresponding zero-dimensional boundary component $\mathcal{F}_0'$ of $\mathcal{D}_L$ under this identification.
Hence: The fan $\Sigma(\mathcal{F}_0')$ constructed in construction 11.2.1 is a rational polyhedral partial decomposition of $\overline{\mathcal{C}(\mathcal{F}_0)}^{\text{rat}}$.
We claim that it is $\overline{\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}}$-admissible:

Lemma 11.2.5. *Let $N \geq 1$ and consider the lattice $II_{2,10}(N)$ with $\Gamma = \widetilde{\mathrm{SO}}^+(II_{2,10}(N))$ and a zero-dimensional boundary component $\mathcal{F}_0$ of $\mathcal{D}_{II_{2,10}(N)} \cong \mathcal{D}_{II_{2,10}}$. Let $\Sigma(\mathcal{F}_0')$ be defined as in construction 11.2.1 for $\mathcal{F}_0'$ the boundary component of $\mathcal{D}_{II_{2,10}}$ corresponding to $\mathcal{F}_0$. The rational polyhedral partial decomposition $\Sigma(\mathcal{F}_0)$ is $\overline{\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}}$-admissible decomposition of $\overline{\mathcal{C}(\mathcal{F}_0)}^{rat}$ and defines a Γ-admissible family.*

Proof. We already showed that $\Sigma(\mathcal{F}_0)$ is a rational partial polyhedral decomposition of the correct cone. We note that $\overline{\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}} = \widetilde{\mathrm{SO}}^+(II_{1,9}(N)) \subseteq \mathrm{SO}^+(II_{1,9}(N))$ and $\Sigma(\mathcal{F}_0)$ is by construction even invariant with respect to the larger group. Moreover: The index $[\mathrm{SO}^+(II_{1,9}) : \widetilde{\mathrm{SO}}^+(II_{1,9}(N))]$ is finite and the number of $\mathrm{SO}^+(II_{1,9})$-orbits of cones in $\Sigma(\mathcal{F} - 0)$ is finite, so there are also only finitely many $\overline{\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}}$-orbits in $\Sigma(\mathcal{F}_0)$. The Γ-admissible family is the constructed as in construction 8.2.1 and the following remark. □

We call this $\widetilde{\mathrm{SO}}^+(II_{2,10}(N))$-admissible family again *Coxeter family* and the resulting toroidal compactification of $X(II_{2,10}(N))$ the *reflective toroidal compactification.*
We note:

Proposition 11.2.6. *The Coxeter family is smooth. If $\Gamma = \widetilde{\mathrm{SO}}^+(II_{2,10}(N))$ is neat, the reflective toroidal compactification is smooth as well.*

Proof. By proposition 8.2.5 the latter part follows from the former and it suffices to check smoothness on $\Sigma(\mathcal{F}_0)$ for a single zero-dimensional $\mathcal{F}_0$ and even for the faces of a fundamental Weyl chamber. By example 11.1.14, any of these is spanned by fundamental weights $w_i \in II_{1,9}$ which are part of a basis of $II_{1,9}$, but then the same cone is spanned by $\sqrt{N}w_i \in \sqrt{N}II_{2,10}$, which constitute a basis of $II_{1,9}(N) \cong \sqrt{N}II_{1,9}$ inside $\mathcal{U}(\mathcal{F}_0)$ via the identification $\mathcal{D}_{II_{2,10}(N)} \cong \mathcal{D}_{II_{2,10}}$. □

The next proposition shows that the reflective toroidal compactification is also projective. We will need the following technical lemma to prove it:

Lemma 11.2.7. *Let $X \subseteq \mathbb{R}^n$ be a convex set. A homogeneous function $f : X \to \mathbb{R}^+$ is concave if the set $\{x \in X | f(x) \geq 1\}$ is convex.*

Proof. Let $x_1, x_2 \in X$ and $\alpha \in (0,1)$. Suppose $f(x_1) \geq f(x_2)$ with $f(x_1) = \frac{1}{s}f(x_2)$ for some $s < 1$. Write

$$\alpha = \frac{st}{1 - t + st}$$

for some $t \in \mathbb{R}$ (this is always possible). We note that, for every $t \in (0,1)$, we have

$$f(tsx_1 + (1-t)x_2) \geq tf(sx_1) + (1-t)f(x_2) = tsf(x_1) + (1-t)f(x_2)$$

by homogeneity of f and the convexity of $\{x \in X | f(x) \geq 1\}$. We have

$$\begin{aligned} f\left(\alpha x_1 + (1-\alpha)x_2\right) &= f\left(\frac{st}{1-t+st}x_1 + \frac{1-t}{1-s+st}x_2\right) \\ &= \frac{1}{1-t+st}f(tsx_1 + (1-t)x_2) \\ &\geq \frac{1}{1-t+st}\left(tsf(x_1) + (1-t)f(x_2)\right) \\ &= \alpha f(x_1) + (1-\alpha)f(x_2) \end{aligned}$$

and see that f is concave. □

With this we can prove the projectivity of the reflective compactification:

Proposition 11.2.8. *The reflective toroidal compactification induced by the Coxeter family is projective.*

Proof. Let $\mathcal{F}_0$ be a zero-dimensional boundary component of $\mathcal{D}_{II_{2,10}(N)}$ and let $\overline{C_0}$ be a fundamental Weyl chamber for the action of $\mathrm{O}^+(II_{2,10})$ as before (or, equivalently, a full-dimensional cone in $\Sigma(\mathcal{F}_0)$). Suppose there is a functional $g : \mathcal{U}(\mathcal{F}_0) \to \mathbb{R}$ with

(i) $g(w_i) \in \frac{1}{\sqrt{N}}\mathbb{N}_{>0}$ for any fundamental weight $w_i \in II_{1,9}$ and

(ii) $g(\rho_i) < 0$ for any fundamental root $\rho_i \in II_{1,9}$

then we can define a function $\phi(\mathcal{F}_0) : \mathcal{C}(\mathcal{F}_0) \to \mathbb{R}$ with the desired properties as follows: On $\overline{C_0}$, set $\phi(\mathcal{F}_0)|_{\overline{C_0}} = g|_{\overline{C_0}}$ and on any $\mathrm{O}^+(II_{2,10})$-image $\gamma\overline{C_0}$ set

$$\phi(\mathcal{F}_0)|_{\gamma\overline{C_0}} = g|_{\overline{C_0}} \circ \gamma^{-1}.$$

Note that this is well-defined, since $\overline{C_0}$ is a strict fundamental domain for the action of $\mathrm{O}^+(II_{1,9})$ on $\mathcal{C}(\mathcal{F}_0)$. By construction, any ϕ_F is $\overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$-invariant (even $\mathrm{O}^+(II_{1,9}) \supset \overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$-invariant), piecewise linear and continuous (the definition for neighboring chambers agree on the walls between them) as well as positive; furthermore it is integral on $\mathcal{U}(\mathcal{F}) = \sqrt{N}II_{1,9}$ in $\overline{\mathcal{C}(\mathcal{F})}^{\mathrm{rat}}$ as it is spanned by the $\sqrt{N}$-rescalings of fundamental weights w.r.t. $II_{1,9}$ and their $\mathrm{O}^+(II_{1,9})$-images.
We check the remaining convexity and linearity properties (note that these are independent from the integral structure, so we assume $N = 1$ for better readability): By lemma 11.2.7 it suffices for having convexity to prove that

$$M = \{x \in \overline{\mathcal{C}(F_0)}^{\mathrm{rat}} | \phi(\mathcal{F}_0)(x) \geq 1\}$$

is convex. First of all, we note the following property of g: For all $x \in \overline{C_0}$ and automorphisms $\gamma \in \mathrm{O}^+(II_{1,9})$ we have

$$g(\gamma x) \geq g(x).$$

We prove this via induction on the length of γ. For $l(\gamma) = 1$ we have $\gamma = \sigma_i$, the reflection corresponding to some fundamental root ρ_i:

$$\begin{aligned} g(\sigma_{\rho_i} x) &= g\left(x - 2\frac{(x,\rho_i)}{(\rho_i,\rho_i)}\rho_i\right) \\ &= g(x) - \underbrace{(x,\rho_i)}_{\geq 0}\underbrace{g(\rho_i)}_{\leq 0} \\ &\geq g(x) \end{aligned}$$

Suppose that $l(\gamma) = n+1$ and the result is true up to n: We write $\gamma = \sigma_j\gamma'$ with σ_j a reflection corresponding to a fundamental root and γ' with $l(\gamma') = n$. By lemma 11.1.9 we have

$$\gamma'\,\overline{C_0} \subseteq H^+_{\sigma_j},$$

so $(\gamma' x, \rho_j) \geq 0$. This allows us to see

$$\begin{aligned} g(\gamma x) &= g\left(\sigma_j\gamma' x\right) \\ &= g\left(\gamma' x - (\gamma' x, \rho_j)\rho_j\right) \\ &= g(\gamma' x) - \underbrace{(x,\rho_j)}_{\geq 0}\underbrace{g(\rho_j)}_{\leq 0} \\ &\geq g(\gamma' x) \\ &\geq g(x) \end{aligned}$$

by the induction hypothesis. Furthermore, we claim

$$M = \overline{\mathcal{C}(F_0)}^{\mathrm{rat}} \cap \bigcap_{\gamma\in\mathrm{O}^+(II_{1,9})} (g\circ\gamma)^{-1}([1,\infty]),$$

which is convex as the intersection of the convex cone $\overline{\mathcal{C}(F_0)}^{\mathrm{rat}}$ with half-spaces: For $x \in M$ we have $x = \gamma_0 x_0$ for some $x_0 \in \overline{C_0}$ and $1 \leq \phi(\mathcal{F}_0)(x) = \phi(\mathcal{F}_0)(\gamma x_0) = g(x_0)$. By the preceding consideration we have

$$(g\circ\gamma)(x) = g(\gamma\gamma_0 x_0) = g(\gamma' x_0) \geq g(x_0) \geq 1.$$

On the other hand, any $x \in \overline{\mathcal{C}(F_0)}^{\mathrm{rat}}$ with $(g\circ\gamma)(x) \geq 1$ is obviously in M. In total, M is convex.
This arguments also implies that the images of $\overline{C_0}$ are the maximal cones on which $\phi(\mathcal{F}_0)$ is linear: Suppose that it is linear on some cone C not contained in a $\gamma\overline{C_0}$, then C has nonempty intersection with a $\gamma\overline{C_0}$ and the image $\sigma\gamma\overline{C_0}$ of $\gamma\overline{C_0}$ under the reflection σ defined by some root ρ. By the $\mathrm{O}^+(II_{1,9})$-invariance of $\phi(\mathcal{F}_0)$ we can assume $\gamma = \mathrm{id}$ and

ρ is a fundamental root ρ_i. For $x \in C \setminus H_\sigma$ (this exists close to the reflection wall) we have $\phi(\mathcal{F}_0)(x) = \phi(\mathcal{F}_0)(\sigma x)$ by definition, but linearity on C would imply

$$\phi(\mathcal{F}_0)(\sigma x) = g(x - (x,\rho_i)\rho_i) = g(x) - \underbrace{(x,\rho_i)}_{>0}\underbrace{g(\rho_i)}_{<0}$$

since x is in the interior of $\overline{C_0}$; this contradicts the linearity. Finally: As $\mathrm{O}^+(II_{2,10})$ acts transitive on the set of zero-dimensional Baily-Borel cusps, we can extend this function $\phi(\mathcal{F}_0)$ canonically to a compatible family $\{\phi(\mathcal{F})\}_{\mathcal{F}}$ in the sense of lemma 3.1.8.
To end this proof we have to guarantee the existence of a functional g with the claimed properties: Set $g(\sqrt{N}\rho_i) \in \mathbb{Z}_{<0}$ arbitrary for the fundamental roots ρ_i w.r.t. $II_{1,9}$ and note

$$w_j = \sum_{i=1}^{n} \lambda_{ij}\rho_i \text{ with } (\lambda_{ij})_{ij} = A^{-1}$$

for A^{-1} the Gram matrix of the fundamental weights from example 11.1.12. Since A^{-1} has only non-positive entries (in fact only $\lambda_{11} = 0$), we see

$$g(w_j) = \sum_{i=1}^{10} \underbrace{\lambda_{ij}}_{\leq 0}\underbrace{g\left(\rho_i\right)}_{<0} > 0$$

and the function $\phi(\mathcal{F}_0)$ defined by this g has all the desired properties. □

We summarize the properties of the reflective toroidal compactification in the following theorem:

Theorem 11.2.9. *Let $II_{2,10}$ the unique even, unimodular lattice of signature $(2,10)$ and let $N \geq 1$. Let $\Gamma = \widetilde{\mathrm{SO}}^+(II_{2,10}(N))$ be the discriminant kernel of $II_{2,10}(N)$. Then:*
The Coxeter family Σ is a smooth and projective Γ-admissible family that defines a toroidal compactification $\overline{X}_\Sigma^{tor}$ with $\partial\overline{X}_\Sigma^{tor}$ a normal crossing divisor.
If Γ is neat, the reflective toroidal compactification $\overline{X}_\Sigma^{tor}$ is a smooth and projective Hausdorff compactification of $X(II_{2,10}(N)) = \Gamma\backslash\mathcal{D}_{II_{2,10}(N)}$ and the boundary is a simple normal crossing divisor.

Proof. The only thing left to prove is the assertion about the boundary being a simple normal crossing divisor. We have to check the criterion of lemma 3.1.10:
By assumption Γ is neat; fix a boundary component $\mathcal{F}$ and let $\gamma \in \Gamma$ and $\sigma \in \Sigma(\mathcal{F})$ with $\gamma(\sigma) \cap \sigma \neq \{0\}$. If $\mathcal{F}$ is a one-dimensional boundary component, the cone $\mathcal{C}(\mathcal{F})$ is isomorphic to $\mathbb{R}_+$ and any γ with $\gamma(\sigma) \cap \gamma \neq \{0\}$ acts as the identity on $\mathcal{C}(\mathcal{F})$. Consider the case of a zero-dimensional boundary component $\mathcal{F}$: We can assume that $\gamma(\sigma) \cap \sigma \neq \{0\}$ happens for $\sigma \preceq \overline{C_0}$, a face of the fundamental Weyl chamber $\overline{C_0}$. Since it is a strict fundamental domain by proposition 11.1.15, we see that γ stabilizes all points of $S = \gamma(\sigma) \cap \sigma$.

We distinguish two cases:

(i) S is contained in the boundary of $\overline{\mathcal{C}(\mathcal{F})}^{\mathrm{rat}}$: Then S is actually the single ray $\mathbb{R}_+ w_0$ spanned by the isotropic fundamental weight w_0 which corresponds an adjacent one-dimensional Baily-Borel cusp $\mathcal{F}_1 \succ \mathcal{F}$. This is just the case we considered before and hence γ acts trivial on $\mathbb{R}_+\rho$ which is the smallest boundary component containing S.

(ii) There exists $x \in S$ in the interior of $\overline{\mathcal{C}(\mathcal{F})}^{\mathrm{rat}}$. By proposition 11.1.15 resp. its proof this stabilizer $\mathrm{Stab}(x)$ is the Coxeter group generated by the reflections S_i in the walls H_i of the fundamental Weyl chamber with $x \in H_i$. If this group is finite, the neatness of Γ implies that γ is already the identity (since neat implies torsion-free). It remains to show that the stabilizer of $x \in \mathcal{C}(\mathcal{F})$ is finite. This is most easily done by checking the Coxeter-Dynkin diagrams of the stabilizer group: The only subdiagram of the Coxeter-Dynkin diagram of E_{10} corresponding to an infinite Coxeter group is the one corresponding to the reflection group of E_9. This occurs as a stabilizer if and only if x is contained in all but a distinguished one of the walls of the fundamental Weyl chamber; this distinguished wall corresponds to the outermost node of the long arm of the Coxeter-Dynkin diagram of E_{10}. In terms of the fundamental weights this condition on x is equivalent to x being in the ray $\mathbb{R}_+ w_0$ spanned by the unique isotropic fundamental weight w_0: however, this cannot happen, since $\mathbb{R}_+ w_0$ lies in the boundary of $\overline{\mathcal{C}(\mathcal{F})}^{\mathrm{rat}}$

□

We briefly consider the singularities $X = X(II_{2,10}(N))$ and its reflective toroidal compactification for non-neat Γ:
As stated before, the toroidal compactification $\overline{X}^{\mathrm{tor}}_{\Sigma}$ cannot be smooth if X already has singularities. We described the case of the rather mild type of canonical singularities in section 8.3, which is the best one can hope for in the non-neat case. As canonicity of singularities is a consequence of the smoothness of Σ for $n \geq 9$, we have for $L = II_{2,10}(N)$ of signature $(2, 10)$:

Lemma 11.2.10. *For $N \geq 1$ and $X = X(II_{2,10}(N))$ the reflective toroidal compactification $\overline{X}^{tor}_{\Sigma}$ has only canonical singularities.*

Even though the construction of the Coxeter family is specifically tailored to the $II_{2,10}$-lattice, some generalization to further lattice seems feasible.

Generalizations

We note the following central ingredient in the construction of the reflective toroidal compactification:
The group $\overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$ appearing as the automorphism group of the cone $\mathcal{C}(\mathcal{F})$ corresponding to a zero-dimensional cusp F is a finite-index subgroup of the hyperbolic reflection group

$W(II_{1,9})$ of $II_{1,9}$ and a fundamental domain of this action of this reflection group has only finitely many walls.
If this finiteness condition holds for a lattice L, an analogous construction yields a reflective compactification of the corresponding locally symmetric space $X(L)$. This should be possible at least for the case of $L = II_{2,18}$ (cf. [Con83]). A zero-dimensional boundary component $\mathcal{F}$ induces again a Siegel domain realization with positive-time-like cone $\mathcal{C}(\mathcal{F})$. Its reflection group has index 2 in the group of autochronous symmetries $\mathrm{O}^+(II_{1,17})$ and there is only one $\mathrm{O}^+(II_{2,18})$-orbit of the zero-dimensional boundary components; a fundamental domain of the action of the reflection group is again a polyhedral cone with finitely many walls, so this defines a Coxeter family, which in turn induces a reflective compactification.
Moreover, Alexeev, Engel and Thompson in [AET19] consider a similar toroidal compactification for the hyperbolic lattice $II_{1,1} \oplus \mathrm{E}_8^2 \oplus A_1$ of signature $(19, 1)$, so there may be a yet-undeveloped more general theory underlying these considerations. We have not looked further into this.
Unfortunately, this approach does not work for the case of $II_{2,26}$: A fundamental domain of the corresponding reflection group has an infinite number of rays (abstractly isomorphic to the Leech lattice); hence the Coxeter family for this lattice is not $\overline{\mathcal{P}(\mathcal{F})}_{\mathbb{Z}}$-finite and a different approach is needed.

11.3. Divisors on the reflective compactification

We want to apply the results of chapter 9 to the reflective compactification and use our more concrete knowledge about this compactification to refine them further.
We start with toroidal boundary divisors of one-dimensional type. Proposition 9.1.4 yields for $L = II_{2,10}(N)$:

Lemma 11.3.1. *Let $N \geq 1$ such that the discriminant kernel Γ of $II_{2,10}(N)$ is neat and let $\overline{X}^{tor}_{\Sigma}$ be the reflective compactification of $X(II_{2,10}(N))$. The toroidal divisor of one-dimensional type over a Baily-Borel cusp $F_1 = \Gamma_1\backslash\mathbb{H}$ is the toroidal compactification of the Kuga-Sato variety $K^{\mathrm{E}_8}_{\Gamma(N)}$ as in section 3.3.*
This toroidal compactification of $K^{\mathrm{E}_8}_{\Gamma(N)}$ is defined by the collection of fans induced by the Coxeter fan in the following way: For any cusp $F_0 \preceq F_1$ the cone decomposition is given by the star $\mathrm{Star}_{\Sigma(\mathcal{F}_0)}(\mathcal{C}(\mathcal{F}_1))$ of $\mathcal{C}(\mathcal{F}_1)$ in $\Sigma(\mathcal{F}_0)$.

The number of these divisors is equal to the number ν_1 of one-dimensional Baily-Borel cusps which can be computed as in proposition 6.2.6.
We note that toroidal divisors of one-dimensional type are discrete on $\overline{X}^{\mathrm{tor}}_{\Sigma}$ in the following sense:

Lemma 11.3.2. *Let $D_1 \neq D_2$ be toroidal boundary divisors of the one-dimensional type in $\overline{X}^{tor}_{\Sigma}$, then*
$$D_1 \cap D_2 = \emptyset.$$

Proof. Consider the morphism $\pi : \overline{X}^{\text{tor}}_{\Sigma} \to \overline{X}^{\text{BB}}$. For a given one-dimensional Baily-Borel cusp F_i there is exactly one toroidal boundary divisor D_i of one-dimensional type with $\pi(D_i) \subseteq \overline{F}_i$ and hence

$$\pi(D_1 \cap D_2) \subseteq \pi(D_1) \cap \pi(D_2) = \overline{F_1} \cap \overline{F_2},$$

so the intersection is over a zero-dimensional Baily-Borel cusp F_0 which is on the common boundary of F_1 and F_2.
In this case, D_1 and D_2 correspond to *different* $\overline{\mathcal{P}(F_0)}_{\mathbb{Z}}$-orbits of isotropic rays in the admissible cone decomposition $\Sigma(\mathcal{F}_0)$ and by construction of the toroidal compactification it is clear that corresponding toroidal boundary divisors intersect if and only if there is $\sigma \in \Sigma_{F_0}$ with $\rho_1, \rho_2 \preceq \sigma$ for some representatives ρ_1, ρ_2 of the orbits $[\rho_1], [\rho_2]$. By construction of the fan defining the compactification of construction 11.2.1, this cannot happen as every cone of $\Sigma(\mathcal{F}_0)$ contains at most one isotropic ray. □

We turn to the toroidal boundary divisor of zero-dimensional type. The theorem 11.2.9 of the last section showed that the reflective compactifications satisfies the condition in lemma 3.1.10 for suitable N, so lemma 9.2.3 shows:

Lemma 11.3.3. *Let $N \geq 1$ such that the discriminant kernel Γ of $II_{2,10}(N)$ is neat and let $\overline{X}^{tor}_{\Sigma}$ be the reflective compactification of $X(II_{2,10}(N))$, then the toroidal boundary divisor of zero-dimensional type $\overline{D}$ are the toric varieties of the form*

$$D_{F_0,\rho} := X_{\text{Star}_{\Sigma(\mathcal{F}_0)}(\rho)}$$

with ρ a non-isotropic ray in Σ_{F_0} and F_0 running through the zero-dimensional Baily-Borel cusps of $X(II_{2,10}(N))$.

We can say even more about these toric varieties:

Proposition 11.3.4. *The toric varieties $D_{F_0,\rho}$ are smooth and compact.*

Proof. We are working over a fixed zero-dimensional Baily-Borel cusp F_0 so we fix a corresponding rational zero-dimensional boundary component $\mathcal{F}_0$ of $\mathcal{D}_L$. With this, we have fixed the usual choice of coordinates on $II_{2,10}(N)$ with the associated lattice K.
We have to check that the fan $\text{Star}_{\Sigma(\mathcal{F}_0)}(\rho)$ satisfies the conditions in lemma 2.1.9 and lemma 2.1.10. Consider the projection map $p : K \to K/(K \cap \rho) = K/\mathbb{Z}\tau$, where τ is the primitive generator of ρ.
Let $\overline{\sigma} \in \text{Star}_{\Sigma(\mathcal{F}_0)}(\rho)$, then there is $\sigma \in \Sigma(\mathcal{F}_0)$ with $\rho \preceq \sigma$ and $p(\sigma) = \overline{\sigma}$. Since σ is generated by parts of a basis of K of the form $\tau, w_1, \ldots, w_{n-1}$ (it is the automorphic image of a face of the fundamental Weyl chamber, which has this property), we see that $\overline{\sigma}$ is generated by the vectors $p(w_1), \ldots, p(w_{n-1})$ which constitute a basis of $K/\mathbb{Z}\tau$, hence any cone in $X_{\text{Star}_{\Sigma(\mathcal{F}_0)}(\rho)}$ is smooth.
We note that the fan $X_{\text{Star}_{\Sigma(\mathcal{F}_0)}(\rho)}$ is finite if and only if there are only finitely many $\sigma \in \Sigma(\mathcal{F}_0)$ with $\rho \preceq \sigma$. By construction of the fan $\Sigma(\mathcal{F}_0)$ we can assume that ρ is a ray in the interior of the fundamental Weyl chamber $\overline{C_0}$.

Suppose there are infinitely many $\sigma \in \Sigma_{\mathcal{F}_0}$ with $\tau \preceq \sigma$, then at least two of them are in the same $\overline{\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}}$-orbit as their number is finite by the properties of an admissible family. In particular, there are $\gamma_1 \neq \gamma_2 \in \overline{\mathcal{P}(\mathcal{F}_0)_{\mathbb{Z}}}$ and $\sigma \preceq \sigma_0$ with $\tau \preceq \gamma_1\sigma$ and $\tau \preceq \gamma_2\sigma$. As usual we observe

$$\gamma\sigma \cap \sigma \neq \{0\}$$

for $\gamma = \gamma_2^{-1}\gamma_1$ and hence $\gamma \in \mathrm{Stab}(\gamma_2^{-1}\tau)$. Since $\gamma_2^{-1}\tau$ is again a non-boundary ray and hence definite as a subspace, its stabilizer is finite, therefore trivial by neatness. We get $\gamma_1 = \gamma_2$ in contradiction to the assumption. This shows the finiteness of $X_{\mathrm{Star}_{\Sigma(\mathcal{F}_0)}(\rho)}$.
Lastly, we prove that the support $|\mathrm{Star}_{\Sigma_(F_0)}(\rho)|$ is $(K/\mathbb{Z}\tau)_{\mathbb{R}}$. As before, denote the primitive generator of ρ by τ and by T the set of full-dimensional cones $\sigma \in \Sigma$ with $\rho \preceq \sigma$. We note that, by construction, any cone in Σ is the face of a full dimensional cone, so, for any of the (finitely many) full-dimensional cones $\sigma_1, \ldots, \sigma_m$ there is exactly one face H_i of codimension 1 not containing ρ. Define

$$\epsilon_\rho = \min \{d(\tau, H_i) |\, i = 1, \ldots, m\}$$

the minimal distance of τ to one of the walls. We have that the ϵ-ball around τ is contained in the union of the cones in T, so

$$B_{\epsilon_\rho}(\tau) \subseteq \bigcup_{\sigma \in T} \sigma$$

and by homogeneity

$$B_{R\epsilon_\rho}(R\tau) \subseteq \bigcup_{\sigma \in T} \sigma$$

for any $R > 0$. Let now $\overline{x_0} = x_0 + \mathbb{R}\tau \in (K/\mathbb{Z}\tau)_{\mathbb{R}}$ be arbitrary. For $R_0 > d(x_0, 0)/\epsilon_\rho$ we see that

$$d(x_0 + R_0\tau) = d(x_0, 0) < R_0\epsilon_\tau$$

and therefore

$$x_0 + R_0\tau \in B_{R_0\epsilon_\rho}(R\tau) \subseteq \bigcup_{\sigma \in T} \sigma.$$

Applying the projection map $p_{\mathbb{R}} : K_{\mathbb{R}} \to K_{\mathbb{R}}/\rho$ show then

$$[x_0] = p(x_0 + R_0\tau) \subseteq p\left(\bigcup_{\sigma \in T} \sigma\right) = \bigcup_{\overline{\sigma} \in \mathrm{Star}_{\Sigma_{F_0}}(\rho)} \overline{\sigma} = |\mathrm{Star}_{\Sigma(F_0)}(\rho)|,$$

so the fan $\mathrm{Star}_{\Sigma(F_0})(\rho)$ is indeed complete. □

Note that the last results are exactly the assumptions in lemma 10.2.2.
For later reference we also explicitly state the exact number of toroidal boundary divisors of zero-dimensional type:

Lemma 11.3.5. *Fix a fundamental Weyl chamber $\overline{C_0}$ and denote the set of its rays by $\mathbb{R}$. The number of toroidal boundary divisors of zero-dimensional type on the reflective compactification $\overline{X}_\Sigma^{tor}$ is exactly*

$$\nu_0 \cdot \sum_{\rho \in \mathbb{R}} \left[\mathrm{O}^+(I\!I_{1,9}(N)) : \mathrm{Stab}_{\widetilde{\mathrm{SO}}^+(I\!I_{1,9}(N))}(\rho) \right],$$

with ν_0 the number of zero-dimensional Baily-Borel cusps of X, computable as in proposition 6.2.6.

Proof. The origin of ν_0 in this formula is clear. For fixed zero-dimensional Baily-Borel cusp F_0 any non-isotropic ray of $\Sigma(\mathcal{F}_0)$ gives rise to a divisor in the partial compactification corresponding to $\mathcal{F}_0$. The gluing procedure identifies all of the divisors corresponding to non-isotropic rays in the same $\overline{\mathcal{P}(\mathcal{F}_0)_\mathbb{Z}}$-orbit with each other, so their number is equal to the number of the orbits, as claimed. $\square$

This is compatible with the analogous result in [YZ14, Theorem 2.20] for the Siegel case. This finishes our closer treatment of toroidal boundary divisors of the reflective compactification. We turn to a certain case of special divisors in the next section.

Special divisors for $\mathbf{I\!I_{2,10}(N)}$

We recall the general idea of section 9.3: The irreducible components of Heegner divisors, the special divisors, can be identified as embeddings of lower-dimensional locally symmetric spaces $X(K_\lambda)$ into $X(L)$. The central result of proposition 9.3.2 can be applied very successfully for our case of $L = I\!I_{2,10}(N)$ for N sufficiently large.
Let $\lambda \in L$ primitive with corresponding lattice $K_\lambda = K \cap \lambda^\perp$ and discriminant kernel $\Gamma(K_\lambda)$. Then:

Proposition 11.3.6. *The lower horizontal map in the diagram*

$$\begin{tikzcd} \mathcal{D}_{K_\lambda} \arrow[r, hook] \arrow[d, "\pi_{\Gamma(K_\lambda)}"] & \mathcal{D}_L \arrow[d, "\pi_{\Gamma(L)}"] \\ \Gamma(K_\lambda)\backslash\mathcal{D}_{K_\lambda} \arrow[r, hook] & \Gamma(L)\backslash\mathcal{D}_L \end{tikzcd}$$

is a closed immersion.

Assume that $K_\lambda = K_\nu$ for some primitive $\nu \in L$ with $q(\nu) = -2N$. We want to apply the results of theorem 3.2.4 and extend this embedding to the reflective toroidal compactification.
Let F_1 be a cusp of $X(K_\lambda)$. By lemma 3.2.1 there exists a unique cusp F_2 of $X(L)$ corresponding to L with $\psi_{\mathrm{BB}}(F_1) \subseteq F_2$.
We can see this explicitly as follows: The cusp F_1 is defined by an $\Gamma(K_\lambda)$-equivalence class of isotropic lattices I of rank i for $i = 1, 2$ with $I \subseteq K_\lambda \subseteq L$ and $\psi_{\mathrm{BB}}(F_1) \subseteq F_2$

means that we can choose $I \subseteq K_\lambda \subseteq L$ defining a common preimage $\mathcal{F} \subseteq \overline{\mathcal{D}_{K_\lambda}} \subseteq \overline{\mathcal{D}_L}$ of the F_i under the projection to the locally symmetric spaces.
We can consider the realizations as Siegel domains of the third kind of the symmetric spaces with respect to this rational boundary component. One can show that the objects corresponding to $\mathcal{D}_{K_\lambda}$ can be obtained by intersecting those for $\mathcal{D}_L$ with $\lambda^\perp$. To be more precise: Let $I \subseteq K_\lambda \subseteq L$ be isotropic of rank 1, generated by $e \in K_\lambda$ and consider the splitting $L = \langle e, e' \rangle \oplus II_{1,9}(N)$ with $e' \in K'_\lambda$ as in section 6.3. We note $\lambda \in II_{1,9}(N) \otimes \mathbb{R}$ by construction.
The image $\iota(\mathcal{C}(F_1))$ of the cone $\mathcal{C}(F_1)$ in $\mathcal{C}(F_2)$ under the map $\iota : \mathcal{C}(F_1) \hookrightarrow \mathcal{C}(F_2)$ as in lemma 3.2.1 is just given by

$$\iota(\mathcal{C}(F_1)) = \mathcal{C}(F_2) \cap \lambda^\perp.$$

The same works for $I \subseteq K_\lambda \subseteq L$ isotropic of rank 2: In this case one even gets

$$\iota(\mathcal{C}(F_1)) = \mathcal{C}(F_2) \cap \lambda^\perp = \mathcal{C}(F_2).$$

Let now Σ be the Coxeter family. The situation is as in construction 11.2.1 and by proposition 3.2.2 the Coxeter family Σ induces a $\Gamma(K_\lambda)$-admissible family Σ_λ for $X(K_\lambda)$.
The central point now is the following: Our special choices of Σ as the Coxeter family and λ as being a multiple of a root of $II_{2,10}(N)$ result in the two families being strictly compatible.

Proposition 11.3.7. *The admissible families Σ_λ and Σ are strictly compatible.*

Proof. As usual, there is nothing to show over one-dimensional cusps, so we just have to consider what happens over zero-dimensional cusps.
To ease notation, all objects related to $X(K_\lambda)$ will be denoted with the index 1 while those related to $X(L)$ acquire the index 2; so, for example, we have $\Sigma_1 = \Sigma_\lambda$ and $\Sigma_2 = \Sigma$. Let F_2 be any zero-dimensional cusp of $\mathcal{D}_2$ and let F_1 a cusp of $\mathcal{D}_1$ such that $\psi_{\mathrm{BB}}(F_1) \subseteq F_2$. We have

$$\iota(\mathcal{C}(F_1)) = \mathcal{C}(F_2) \cap \lambda^\perp$$

and therefore

$$\Sigma(F_1) = \left\{ \sigma \cap \lambda^\perp \,|\, \sigma \in \Sigma(F_2) \right\}.$$

Since every $\sigma \in \Sigma(F_2)$ satisfies $\sigma \subseteq \overline{\mathcal{C}(F_2)}$ it suffices to show that $\overline{\mathcal{C}(F_2)} \cap \lambda^\perp$ is a cone in $\Sigma(F_2)$ as this decomposition is closed under intersections.
But this is easy to see: We recall that $\Sigma(F_2)$ is defined via the decomposition of the Tits cone $\overline{\mathcal{C}(F_2)}$ (cf. construction 11.2.1) by the walls orthogonal to the roots of $II_{1,9}$ or $II_{1,9}(N)$. By assumption on λ, the hyperplane $\lambda^\perp$ is of the form

$$\lambda^\perp = \nu^\perp,$$

for some primitive $\nu \in L$ with $q(\nu) = -2N$. This is a root of $II_{1,9}(N)$, so we see that $\lambda^\perp \cap \overline{\mathcal{C}(F_2)}$ is a cone in $\Sigma(F_2)$.
In total, this proves the strict compatibility of the two admissible families. $\square$

In the future, we will often neglect the map ψ_{BB} and think of $\psi_{BB}(F_1) \subseteq F_2$ as $F_1 \subseteq F_2$. By proposition 3.2.5, the resulting admissible family Σ_λ is smooth and projective as well. This allows us to identify the closure of $X(K_\lambda)$ in $\overline{X(L)}_\Sigma^{\text{tor}}$ with the toroidal compactification $\overline{X(K_\lambda)}_{\Sigma_\lambda}^{\text{tor}}$ by the induced family Σ_λ.

Theorem 11.3.8. *With the notations as before and* $c_\Sigma : X(L) \hookrightarrow \overline{X(L)}_\Sigma^{tor}$ *the inclusion of* $X(L)$ *into the toroidal compactification satisfies*

$$\overline{c_\Sigma(\psi(X(K_\lambda)))} \cong \overline{X(K_\lambda)}_{\Sigma_\lambda}^{tor}$$

as smooth complex algebraic varieties.

Proof. By theorem 3.2.4 we have a commutative diagram

$$\begin{array}{ccc} X(K_\lambda) & \xrightarrow{\psi} & X(L) \\ \Big\downarrow{\scriptstyle c_{\Sigma_\lambda}} & & \Big\downarrow{\scriptstyle c_\Sigma} \\ \overline{X(K_\lambda)}_{\Sigma_\lambda}^{\text{tor}} & \xrightarrow{\psi_{\text{tor}}} & \overline{X(L)}_\Sigma^{\text{tor}} \end{array}$$

of closed embeddings, so we have

$$\begin{aligned} \overline{c_\Sigma(\psi(X(K_\lambda)))} &= \overline{\psi_{\text{tor}}(c_{\Sigma_\lambda}(X(K_\lambda)))} \\ &= \psi_{\text{tor}}\left(\overline{c_{\Sigma_\lambda}(X(K_\lambda))}\right) \\ &= \psi_{\text{tor}}\left(\overline{X(K_\lambda)}_{\Sigma_\lambda}^{\text{tor}}\right) \\ &\cong \overline{X(K_\lambda)}_{\Sigma_\lambda}^{\text{tor}} \end{aligned}$$

as claimed. □

This will be of tremendous importance in the intersection theory of toroidal boundary divisors of $\overline{X(L)}^{\text{tor}}$ later on.
This shows that certain special divisor can themselves be considered as toroidal compactifications of orthogonal locally symmetric spaces.
To go full circle back to Heegner divisors, it remains to give a criterion for a Heegner divisor to consist only of special divisors as just described. We assume N to be odd at this point.

Lemma 11.3.9. *Let* $H(\beta, m)$ *be a Heegner divisor for* $m \in \mathbb{Q}$ *and* $\beta \in L'/L$*. If*

$$m = -2/N \text{ or } m = -2N$$

then every irreducible component of $H(\beta, m)$ *is of the form* $X(K_\lambda) = \Gamma(K_\lambda)\backslash\mathcal{D}_{K_\lambda}$ *with* $K_\lambda = K_\nu$ *for some primitive* $\nu \in L$ *with* $q(\nu) = -2N$*, that is, a root of* $L = II_{2,10}(N)$*.*

Proof. Let $\beta = \lambda + L$ for some $\lambda \in L'$ with $q(\lambda) = -2/N$. Every non-trivial element of L'/L has an order dividing N, so $N\lambda \in L$. Let $E = \Gamma\backslash\lambda^\perp$ be the associated irreducible component of $H(\beta, m)$, then, by lemma 9.3.7,

$$E = \Gamma(K_\mu)\backslash\mathcal{D}_{K_\mu}$$

for the primitive generator $\mu \in L$ of $\mathbb{Q}\lambda \cap L$ with $q(\mu) = Nc$ for some $c \in \mathbb{Z}$. Note that

$$\lambda = \frac{k}{N}\mu$$

for some $k \in \mathbb{N}$ and

$$Nc = q(\mu) = \frac{N^2}{k^2}q(\lambda) = \frac{-2N}{k^2},$$

so $\frac{-2}{k^2} = c \in \mathbb{Z}$ which implies $k = \pm 1$ and thus $q(\mu) = -2N$.
The computation for $q(\lambda) = -2N$ is similar. $\square$

There is a more general theory underlying these divisors:
For an orthogonal Shimura variety $\Gamma\backslash X$ corresponding to a lattice L of signature $(2, n)$, a divisor of the form $\Gamma\lambda^\perp$ is called a *reflective divisor* if $\Gamma\lambda^\perp$ can be realized as the orthogonal complement of a root $\rho \in L$; the divisor is then said to be *supported on the root* ρ.
This definition for divisors coincides with the characterization given above as the roots of $II_{2,10}(N)$ are just the vectors of norm $-2N$ and the preceding computation shows that the irreducible components of a Heegner divisor of a class $\gamma \in L'/L$ with $q(\gamma) = -2/N$ or $-2N$ are supported on roots.
The following is a summary of the results of the three preceding sections describing the geometry of boundary and special divisors of the reflective toroidal compactification of $X(II_{2,10}(N))$.

Theorem 11.3.10. *If the underlying Baily-Borel cusp F is zero-dimensional, a toroidal boundary divisor D of the reflective toroidal compactification $\overline{X}_\Sigma^{tor}$ is isomorphic to a smooth compact toric variety*

$$X_{\mathrm{Star}_{\Sigma(F)}(\rho)}$$

for some non-isotropic ray $\rho \in \Sigma(F)$; if F is one-dimensional, the divisor is a smooth toroidal compactification of the Kuga-Sato variety

$$\overline{\mathcal{E}^{(n-2)}} = \overline{K_{\Gamma(N)}^{\mathrm{E}_8}}$$

of rank $n-2$ over the modular curve F, the compactification (locally at a cusp F' of F) defined by the star of the orbit of the isotropic ray defining F'.
For any $\lambda \in L$ with $q(\lambda) = -2N$, the natural morphism $X(L \cap \lambda^\perp) \to X(L)$ is a closed immersion and the closure $\overline{X(L \cap \lambda^\perp)} \subseteq \overline{X}_\Sigma^{tor}$ is itself a smooth projective toroidal compactification defined by the $\Gamma(L\cap\lambda^\perp)$-admissible family of cone decompositions $\Sigma_{L\cap\lambda^\perp}$ obtained by restriction.

Higher special cycles

One observes that the construction and the proof of the nice properties of the closure of reflective special divisors can be generalized to chains of lattice inclusions cut out by norm $-2N$-vectors:

Proposition 11.3.11. *Let $L_0 = II_{2,10}(N)$. Define inductively*

$$L_i = L_{i-1} \cap \lambda_i{}^{\perp}$$

by choosing $\lambda_i \in L_{i-1}$ of norm $-2N$.
Then: We have a closed immersion

$$X(L_i) \hookrightarrow X(L_{i-1})$$

and induced admissible families Σ_{L_i}. The closure

$$\overline{X(L_i)} \subseteq \overline{X(L_{i-1})}^{tor}_{\Sigma_{L_{i-1}}}$$

is the toroidal compactification

$$\overline{X(L_i)}^{tor}_{\Sigma_{L_i}}$$

of the orthogonal locally symmetric space $X(L_i)$ of signature $(2, n-i)$ defined by the fan Σ_{L_i} induced by the Coxeter family Σ via the chain

$$X(L_i) \subseteq X(L_{i-1}) \subseteq \ldots \subseteq X(L_0) = X(II_{2,10}(N))$$

as described before in proposition 11.3.7. Since $\overline{X}^{tor}_{\Sigma}$ is smooth and projective, the spaces $\overline{X(L_i)}^{tor}_{\Sigma_{L_i}}$ inherit these properties.

As all of these compactifications ultimately depend only on the choice of the Coxeter family as an input for the toroidal compactification of $X(L_0)$, we will call these *reflective compactifications* as well.

Proof. The existence of the closed immersion is just proposition 9.3.2 applied to

$$L_i = L_{i-1} \cap \lambda_{i-1}{}^{\perp}.$$

The remaining claims follow from the fact that the admissible families Σ_{L_i} and $\Sigma_{L_{i-1}}$ are strictly compatible: The statement about the closure is just theorem 11.3.8 and the statement about smoothness and projectivity from proposition 3.2.5.
As always strict compatibility is trivial over one-dimensional cusps. For the remaining case, the proof is analogous to the one we gave before: Let F_i be a zero-dimensional cusp of the symmetric space associated to the lattice L_i and F_{i-1} the corresponding cusp for L_{i-1}, then, as in proposition 11.3.7,

$$\mathcal{C}(F_i) = \mathcal{C}(F_{i-1}) \cap \lambda_{i-1}{}^{\perp}$$

and

$$\Sigma(F_i) = \Sigma(F_{i-1}) \cap {\lambda_{i-1}}^{\perp}$$

in the obvious notation. By induction it suffices again to show that

$${\lambda_{i-1}}^{\perp} \cap \overline{\mathcal{C}(F_{i-1})} \in \Sigma(F_{i-1}).$$

We compute

$$\begin{aligned}
{\lambda_{i-1}}^{\perp} \cap \overline{\mathcal{C}(F_{i-1})} &= {\lambda_{i-1}}^{\perp} \cap \left({\lambda_0}^{\perp} \cap \ldots \cap {\lambda_{i-2}}^{\perp}\right) \cap \overline{\mathcal{C}(F_0)} \\
&= \left({\lambda_{i-1}}^{\perp} \cap \overline{\mathcal{C}(F_0)}\right) \cap \left({\lambda_0}^{\perp} \cap \ldots \cap {\lambda_{i-2}}^{\perp}\right) \\
&\in \Sigma(F_0) \cap \left({\lambda_0}^{\perp} \cap \ldots \cap {\lambda_{i-2}}^{\perp}\right) \\
&= \Sigma(F_{i-1}),
\end{aligned}$$

noting ${\lambda_{i-1}}^{\perp} \cap \overline{\mathcal{C}(F_0)} \in \Sigma(F_0)$ as in proposition 11.3.7 and

$$\overline{\mathcal{C}(F_{i-1})} = \overline{\mathcal{C}(F_0)} \cap \left({\lambda_0}^{\perp} \cap \ldots \cap {\lambda_{i-2}}^{\perp}\right)$$

by induction. □

One can furthermore see that the toroidal boundary divisors of

$$\overline{X(L_i)}^{\mathrm{tor}}_{\Sigma_{L_i}}$$

are of the same form and with the same properties as those of $\overline{X}^{\mathrm{tor}}_{\Sigma}$, of course with the suitably adapted ranks and induced admissible families defining them, since our description of the boundary divisors in section 9.1 and section 9.2 did not depend on the actual form of the lattice.

This ends our construction and description of the reflective compactification. The next chapter will treat the relations between the different types of divisor on this compactification and explore their intersection theory.

12. Intersection theory on the reflective compactification

From now on we will solely work with the reflective compactification. Let $N \geq 1$ be such that the discriminant kernel Γ of $II_{2,10}(N)$ is neat and $\overline{X}_{\Sigma}^{\text{tor}}$ is the reflective compactification of $X = X(II_{2,10}(N))$; by theorem 11.2.9, this is a smooth projective variety.
The first section of this chapter will explain how to compute intersection products on toric and Kuga-Sato varieties while the second section examines the intersection theory of certain divisors on the reflective compactifications. The final section will explain the characterization and construction of certain relations in the Chow ring of $\overline{X}_{\Sigma}^{\text{tor}}$ which will be central tools for the computation of the error term in the dimension formula for orthogonal modular forms on $II_{2,10}(N)$ in the next chapters.

12.1. Intersection theory on divisors

We described three types of divisors on toroidal compactifications of orthogonal locally symmetric spaces in section 11.3: The toroidal boundary divisors of zero-dimensional type are smooth compact toric varieties, while the toroidal boundary divisors of one-dimensional type are toroidal compactifications of Kuga-Sato varieties. The intersection theories of divisors on these objects will be a tool in the computation of the error term later on, but they are of independent interest.

Intersection theory on toric varieties

We start with the case of toric varieties, without any restriction to those toric appearing as toroidal boundary divisors. Even though the intersection theory of toric varieties is well-developed for simplicial toric varieties, we will treat only the case of smooth compact toric varieties for simplicity.
The interested reader should consult [CLS11, Chapter 6 and 13] for the general case and Tsushima's work [Tsu80, Section 2], which we will follow closely in this section.
For this section we assume Σ to be a smooth fan in $\mathbb{R}^n = X^*(T) \otimes \mathbb{R}$ and X_Σ the corresponding smooth compact toric variety as in chapter 2. By the orbit-cone correspondence (proposition 2.1.11) a k-dimensional cone $\sigma \in \Sigma$ corresponds to an $n-k$-dimensional torus orbit $O(\sigma)$ in X_Σ.
We will denote the set of one-dimensional cones of Σ by $\Sigma(1)$ and call its members *rays*, and the set of $n-1$-dimensional cones by $\Sigma(n-1)$. These are the so-called *walls*.

The closures $D_\rho = \overline{O(\rho)} \subseteq X_\Sigma$ of the torus orbits $O(\rho)$ for $\rho \in \Sigma(1)$ yield exactly the torus-invariant prime divisors of X_Σ.
The intersection theory of divisors corresponding to pairwise distinct rays

$$\rho_1, \ldots, \rho_d \in \Sigma(1)$$

is as easy as one could hope for:

Proposition 12.1.1 ([CLS11, Lemma 12.5.2])**.** *Let $\rho_1, \ldots, \rho_d \in \Sigma(1)$ be pairwise distinct rays and let $\sigma = \langle \rho_1, \ldots, \rho_d \rangle$ be the cone spanned by these rays. Then*

$$[D_{\rho_1}] \cdot \ldots \cdot [D_{\rho_d}] = \begin{cases} \overline{O(\sigma)} & \textit{if } \sigma \in \Sigma \\ 0 & \textit{else} \end{cases}.$$

In particular

$$[D_{\rho_1}] \cdot \ldots \cdot [D_{\rho_n}] = 1$$

if the rays span a full-dimensional cone of Σ and

$$[D_{\rho_1}] \cdot \ldots \cdot [D_{\rho_n}] = 0$$

if the cone generated by the ρ_i is not in Σ.

This can be generalized to non-pairwise distinct intersection products by the use of *wall relations*. This has a fairly geometric interpretation in terms of the underlying toric geometry.
Let $\rho_1, \rho_{n+1} \in \Sigma(1)$ be rays and $\tau \in \Sigma(n-1)$ be a wall generated by the remaining rays $\rho_2 \ldots, \rho_n \in \Sigma(1)$. Denote by σ resp σ' the cone generated by τ and ρ_1 resp. ρ_{n+1}, then $\tau = \sigma \cap \sigma'$ is a wall separating the two full-dimensional cones σ, σ'. This gives rise to a so-called *wall relation*: Let u_i be the minimal generator of the ray ρ_i, then the $n+1$ vectors $u_1, \ldots, u_{n+1}$ are linearly dependent and we get the relation

$$u_1 + \sum_{i=2}^{n} b_i u_i + u_{n+1} = 0.$$

In this situation we get:

Proposition 12.1.2 ([CLS11, Proposition 6.4.4])**.** *The intersection number of divisors $D_i = D_{\rho_i}$ and the curve $\overline{O(\tau)}$ corresponding to the wall τ is as follows:*

$$[D_{\rho_i}] \cdot \left[\overline{O(\tau)}\right] = \begin{cases} 1 & i \in \{1, n+1\} \\ b_i & i \in \{2, \ldots, n\} \end{cases}$$

This gets more complicated for the intersection of cycles with non-proper intersection of higher dimension. Tsushima in [Tsu80, Section 2] gives an algebraic approach to this problem which we reproduce here. We need a good supply of relations in the Chow ring of X_Σ.
We remember from chapter 2 that the set $X^*(T)$ of characters $T \to \mathbb{C}^*$ forms a free $\mathbb{Z}$-lattice of rank n. Choose a basis of $X_*(T) \otimes \mathbb{R}$ and the corresponding dual basis of $X^*(T) \otimes \mathbb{R}$ with χ^m denoting the character corresponding to $m \in M = X^*(T)$. Then:

Lemma 12.1.3 ([CLS11, Proposition 4.1.2]). *For any $m \in X^*(T)$ we have*

$$\operatorname{div}(\chi^m) = \sum_{\rho\in\Sigma(1)} \nu_{D_\rho}(\chi^m) D_\rho = \sum_{\rho\in\Sigma(1)} \langle m, u_\rho\rangle D_\rho$$

with $\langle\cdot,\cdot\rangle$ the bilinear form from chapter 2 and u_ρ the minimal generator of ρ. In particular

$$0 = \sum_{\rho\in\Sigma(1)} \langle m, u_\rho\rangle\, [D_\rho] \in \mathrm{CH}(X_\Sigma).$$

Tsushima utilizes a special kind of relations induced by the minimal generators of the rays defining the divisors in question, since these yield sufficiently many linear independent relations.

Let

$$[D_{\rho_1}]^{m_1} \cdot \ldots \cdot [D_{\rho_l}]^{m_l} \tag{12.1}$$

be an intersection product and let $\rho_1, \ldots, \rho_{n+m}$ be the rays in $\Sigma(1)$ with minimal generators $u_i = (u_{i,1} \cdots u_{i,n})$. Then:

Corollary 12.1.4 ([Tsu80, Theorem 2.3]). *For every $1 \leq j \leq n$ we have a relation*

$$0 = \sum_{i=1}^{n+m} u_{i,j}\, [D_{\rho_i}] \in \mathrm{CH}(X_\Sigma).$$

These relations are sufficient to compute general intersection products in X_Σ by induction on the difference $n - l$ with l the number of distinct divisors in eq. (12.1).

If $n - l = 0$, then eq. (12.1) is just 1 or 0 by proposition 12.1.1.

Assume now $n - l \leq k$ and assume we computed all the intersection products with at least $n - k = l$ distinct divisors; reorder the factors such that $m_1 \geq 2$. We can assume that the rays $\rho_1, \ldots \rho_l$ defining the divisors are the faces of a full-dimensional cone of Σ (otherwise their product is zero anyway, as the divisors do not have a common intersection).

For any $1 \leq j \leq n$ we compute

$$\begin{aligned} u_{1,j}\, [D_{\rho_1}]^{m_1} \cdot \ldots \cdot [D_{\rho_l}]^{m_l} + \sum_{i=2}^{l} u_{i,j}\, [D_{\rho_1}]^{m_1-1} \cdot \ldots \cdot [D_{\rho_i}]^{m_i+1} \cdot \ldots \cdot [D_{\rho_l}]^{m_l} \\ = (-1) \sum_{i=l+1}^{n+m} u_{i,j}\, [D_{\rho_i}] \cdot [D_{\rho_1}]^{m_1-1} \cdot \ldots \cdot [D_{\rho_l}]^{m_l} =: C_j \end{aligned}$$

by using the j-th relation. The terms on the right hand side of this equation have more than l distinct divisors, so they are already computed by assumption. We note that for any j the intersection numbers

$$[D_{\rho_1}]^{m_1} \cdot \ldots \cdot [D_{\rho_l}]^{m_l}$$

and

$$[D_{\rho_1}]^{m_1-1} \cdot \ldots \cdot [D_{\rho_i}]^{m_i+1} \cdot \ldots \cdot [D_{\rho_l}]^{m_l}$$

are solutions to the linear equation

$$\sum_{i=1}^{l} u_{i,j} X_i = C_j,$$

so they are solutions to the system

$$\begin{pmatrix} u_{1,1} & \dots & u_{l,1} \\ \vdots & \ddots & \vdots \\ u_{1,n} & \dots & u_{l,n} \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ \vdots \\ X_l \end{pmatrix} = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$$

of equations. The matrix has rank l since these are the coordinates of the linear independent minimal generators of the rays $\rho_1, \dots, \rho_l$, so the solution is unique and can be computed by simply solving this system of equations.
This approach can be implemented by computer quite efficiently.

Remark 12.1.5. One can show that proposition 12.1.2 is just an easy special case of the approach just depicted.

Obviously all of this can be applied to those toric varieties appearing as toroidal boundary divisors of zero-dimensional type of the reflective toroidal compactification.

Intersection theory on Kuga-Sato varieties

The second type of toroidal boundary divisors of toroidal compactifications beside the smooth compact toric varieties just treated are the Kuga-Sato varieties which appear as toroidal boundary divisors of one-dimensional type. The computation of intersection numbers of divisors on these varieties is quite similar to the case of toric varieties. We give a short recap of their description, mainly to fix notation:
A toroidal boundary component of one-dimensional type over a Baily-Borel cusp F_1 is isomorphic to a Kuga-Sato variety of $\mathcal{E}^{(8)} \cong K^{\mathrm{E}_8}_{\Gamma(N)}$ of rank 8 over the modular curve $F = \Gamma(N)\backslash\mathbb{H}$ with $\Gamma(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ the integral subgroup $G_{h,\mathcal{P}}(\mathcal{F}_1) \cap \widetilde{\mathrm{SO}}^+(II_{2,10}(N))$ of automorphisms of $\mathbb{H}$ as in section 6.3.
A description of the closure $\overline{\mathcal{E}^{(n-2)}}$ of this boundary component is in section 9.1: Let F_0 be a boundary point of the modular curve F_1, then F_1 corresponds to (the orbit of) an isotropic ray $\rho_0 \in \Sigma(F_0)$.
The part of the closure of $\overline{\mathcal{E}^{(n-2)}}$ lying over F_0 is then given by the quotient of the toric variety $X_{\mathrm{Star}_{\Sigma(F_0)}}(\rho_0)$. The boundary divisors of the compactified Kuga-Sato variety are given by the closures of the boundary components $D_\rho \subseteq X_{\mathrm{Star}_{\Sigma(F_0)}}(\rho_0)$ corresponding to orbits of rays inside $\mathrm{Star}_{\Sigma(F_0)}(\rho_0)$; these are in bijection to the two-dimensional cones of $\Sigma(F_0)$ containing ρ_0 as a face, up to the action of $\overline{\mathcal{P}(F_0)}_{\mathbb{Z}} \cap \overline{\mathcal{P}(F_1)}_{\mathbb{Z}}$.
We make the following observations on intersection products of these divisors:

- If D_1, D_2 are divisors lying over different cusps $F_1, F_2 \subseteq \Gamma(N)\backslash\mathbb{H}$ their intersection is empty by the standard use of the morphism $\pi : \overline{X}^{\mathrm{tor}}_{\Sigma} \to \overline{X}^{\mathrm{BB}}$ as in lemma 11.3.2.

- If D_1, D_2 are divisors both lying over $F_0 \subseteq F_1$, the intersection can be understood locally within $X_{\mathrm{Star}_{\Sigma(F_0)}}(\rho_0)$ by the arguments of remark 4.1.7.

This shows that the intersection theory on Kuga-Sato varieties can be reduced to the intersection theory of toric varieties, and every intersection product of boundary divisor on $\overline{\mathcal{E}^{(n-2)}}$ can be computed by the means of the first part of this section on toric varieties.

12.2. Intersection theory of divisors

We turn our attention to the intersection theory of divisors on the reflective toroidal compactification $\overline{X}_{\Sigma}^{\mathrm{tor}}$ of the orthogonal locally symmetric space $X(II_{2,10}(N))$, so $n = 10$ in all of the following.

Remark 12.2.1. An informal word of caution: As we saw in chapter 9, the objects used to define and describe the boundary components of toroidal compactification usually are orbits of lattice-related objects under the operations of various groups. A typical example is a cusp defined by the Γ-orbit of a primitive isotropic sublattice, or a toroidal boundary divisor of zero-dimensional type, which is defined by the $\overline{\mathcal{P}(F_0)_{\mathbb{Z}}}$-orbit of a ray in an admissible collection of cones in the suitable Siegel domain realization.
To save writing and reading effort, we will adopt a blatant abuse of notation: If we can work near a single Baily-Borel cusp, we will treat these objects as being defined by the lattice-related objects and not their corresponding orbits.
This is justified by the fact that the toroidal compactification near a cusp looks like a quotient (by a properly discontinuous group action cf. [AMRT10, Section III.6]) of the partial compactifications used to define it, so we can restrict our considerations to the partial compactification and use a fundamental domain there.
We will deviate freely from this practice when we deem it useful or necessary to explain the meaning of objects or constructions in question.

12.2.1. Divisors of one-dimensional type

We start with the intersection theory involving boundary divisors of one-dimensional type. For the following, let $D \subseteq \overline{X}_{\Sigma}^{\mathrm{tor}}$ be a fixed toroidal boundary divisor of one-dimensional type and recall its structure as a compactified Kuga-Sato variety as described in proposition 9.1.4.

Intersection with boundary divisors of one-dimensional type

We already proved some part of the intersection theory for toroidal divisors of one-dimensional type, namely $D_i \cap D_j = 0$ for $D_i \neq D_j$ (cf. lemma 11.3.2). In particular, this shows that the intersection product $[D_i] \cdot [D_j]$ of the corresponding cycle classes is zero.
The other possible case of intersection products involving only toroidal boundary divisors of one-dimensional type is that of pure self-intersection: (Self-)Intersection products of the form $[D]^k$ for $k > 1$ are usually handled by the use of relations in the Chow ring:

Since intersection products are defined on the level of cycle classes, we can compute these products by using representatives of $[D]$ whose intersection has a more obvious geometric understanding. By the theory of Borcherds products in section 12.3 we will see that

$$[D] = \sum_j c_j [H_j] + \sum_k c_k^1 [D_k^1] + \sum_l c_l^0 [D_l^0]$$

in the Chow ring of $\overline{X}_\Sigma^{\mathrm{tor}}$, where H_j are special divisors as in section 11.3, D_k^1 are toroidal boundary divisors of one-dimensional type, different from D, and D_l^0 are boundary divisors of zero-dimensional type. In view of this, we can replace

$$\begin{aligned} [D] \cdot [D] &= [D] \cdot \left(\sum_j c_j [H_j] + \sum_k c_k^1 [D_k^1] + \sum_l c_l^0 [D_l^0] \right) \\ &= \sum_j c_j [D] \cdot [H_j] + \sum_k c_k^1 [D] \cdot [D_k^1] + \sum_l c_l^0 [D] \cdot [D_l^0] \end{aligned}$$

and the computation of pure self-intersection products reduces to the computation of intersection products of $[D]$ with special divisors and toroidal boundary divisors of zero-dimensional type, which will be treated in the following paragraphs.

Intersection with special divisors

This section will treat the case of intersection between compactified special divisors and D lying over the one-dimensional Baily-Borel cusp F of $X(L)$.
For this end, fix a special divisor $X(K_\lambda)$ as in section 11.3, that is, an embedded Shimura subvariety of $X(L)$ corresponding to a primitive $\lambda \in L$ of negative norm such that the induced admissible family for $X(K_\lambda)$ is strictly compatible with the one for $X(L)$ and denote its induced toroidal compactification by $\overline{X(K_\lambda)}$ with closed immersion

$$i : \overline{X(K_\lambda)} \hookrightarrow \overline{X(L)}.$$

We assume that F is also a one-dimensional Baily-Borel cusp of $X(K_\lambda)$, i.e. the Γ-orbit of rank-two isotropic lattices defining F has a representative lying inside K_λ. This has several effects: We can choose the coordinates as in section 6.3 compatible in the sense that this lattice induces a decomposition not only of L but also of K_λ. We get a realization of $\mathrm{E}_8(-1)$ in the decomposition $L = II_{1,1} \oplus II_{1,1} \oplus \mathrm{E}_8$ such that $\mathrm{E}_8(-1) \cap \lambda^\perp$ is the negative definite lattice of signature $(0,7)$ appearing in the respective decomposition of K_λ.
The next result will show that the intersection product of $\left[\overline{X(K_\lambda)}\right]$ and $[D]$ in the Chow ring $\mathrm{CH}\left(\overline{X(L)}\right)$ is the class $i_*([D_\lambda])$ of the push-forward of the toroidal boundary divisor D_λ of one-dimensional type in $\overline{X(K_\lambda)}$ over F:

Lemma 12.2.2. *In the situation described above we have*

$$\left[\overline{X(K_\lambda)}\right] \cdot [D] = i_* \left(i^*[D]\right) = i_* \left[D_\lambda\right].$$

The cycle class $[D_\lambda] \in \mathrm{CH}\left(\overline{X(K_\lambda)}\right)$ *can be represented by the toroidal compactification* $\overline{\mathcal{E}^{(n-3)}}$ *of rank* $n-3$ *of the Kuga-Sato variety*

$$\mathcal{E}^{(n-3)} = K_{\Gamma(N)}^{\mathrm{E}_8(-1)\cap\lambda^\perp}$$

over $\overline{\Gamma(N)\backslash\mathbb{H}}$*, with the compactification being the one induced by the closure under*

$$K_{\Gamma(N)}^{\mathrm{E}_8\cap\lambda^\perp} \hookrightarrow \overline{X(K_\lambda)}.$$

Proof. The description of the representative of $[D_\lambda]$ is just the characterization proposition 9.1.4 applied to the toroidal compactification $\overline{X(K_\lambda)}$ of the lattice K_λ of signature $(2, n-1)$. If

$$\operatorname{codim}_{\overline{X(K_\lambda)}}\left(D \cap \overline{X(K_\lambda)}\right) = 1 \left(= \operatorname{codim}_{\overline{X(L)}}(D)\right);$$

we can apply corollary 4.1.5 and are left to prove that

$$D \cap \overline{X(K_\lambda)} = D_\lambda$$

as a scheme-theoretic intersection. We consider the set-theoretic intersection

$$\overline{X(K_\lambda)} \cap D$$

and use the knowledge about the stratification of these objects: The epimorphisms in the diagram

$$\begin{tikzcd} \overline{X(K_\lambda)} \arrow[r, hook] \arrow[d, two heads, "\pi_\lambda"] & \overline{X}_\Sigma^{\mathrm{tor}} \arrow[d, two heads, "\pi_L"] \\ \overline{X(K_\lambda)}^{\mathrm{BB}} \arrow[r, hook] & \overline{X(L)}^{\mathrm{BB}} \end{tikzcd}$$

give us $\overline{X}_\Sigma^{\mathrm{tor}}$ resp. $\overline{X(K_\lambda)}$ as disjoint unions of strata indexed by the Baily-Borel cusps (including the trivial cusp given by the interior) of $\overline{X(L)}^{\mathrm{BB}}$ resp. $\overline{X(K_\lambda)}^{\mathrm{BB}}$. We distinguish the possible pairwise intersections:
As $\pi_L(D) \cap X(K_\lambda) \subseteq \pi_L(D) \cap X(L) \subseteq \partial\overline{X}_\Sigma^{\mathrm{tor}} \cap X(L) = \emptyset$, any intersection between D and the interior of $\overline{X(K_\lambda)}$ is trivial.
For a boundary stratum $S \subseteq \overline{X(K_\lambda)}$ over a cusp F' we have

$$\pi_L(S \cap D) = \pi_L\left(\pi_\lambda^{-1}(F') \cap \pi_L^{-1}\left(\overline{F}\right)\right) \subseteq F' \cap \overline{F},$$

so a boundary stratum of $\overline{X(K_\lambda)}$ meets D only if F' is contained in the closure $\overline{F}$ of F. Again, we distinguish the two possible cases for F':

- $F' = F$ is a one-dimensional cusp: A boundary stratum of one-dimensional type of $\overline{X(K_\lambda)}$ arises as the quotient of (considered in suitable coordinates)

$$\mathbb{H} \times \mathbb{C}^{n-3} = \left(\mathbb{H} \times \mathbb{C}^{n-2}\right) \cap \lambda^\perp$$

by the stabilizer of F' in the discriminant kernel of the lattice K_λ, while D arises in a similar way as a quotient of $\mathbb{H} \times \mathbb{C}^{n-2}$, so the intersection of these strata is the quotient of

$$\mathbb{H} \times \mathbb{C}^{n-3} = \left(\mathbb{H} \times \mathbb{C}^{n-2}\right) \cap \lambda^\perp \subseteq \mathbb{H} \times \mathbb{C}^{n-2},$$

the boundary stratum of one-dimensional type of $\overline{X(K_\lambda)}$ over the cusp F; in particular, it is again an open Kuga-Sato variety, this time of co-rank 2.

- $F' \subseteq \overline{F}$ is a zero-dimensional cusp in the boundary of F: In this case F' is defined by the orbit of an isotropic line in K_λ and $F' \cap \overline{F} = F' \subseteq \partial \overline{F}$, so the intersection $S \cap D$ lies in the boundary of D. The cusp F' of $X(K_\lambda)$ can be considered as a zero-dimensional cusp of $X(L)$ as well. To understand this intersection, we need a better description of the objects at play.

  We chose a fundamental domain $\overline{C_0}$ for the action of $\overline{\mathcal{P}(F')_\mathbb{Z}}$ on the cone $\mathcal{C}(F')$, so we can work interchangeably with rays and their $\overline{\mathcal{P}(F')_\mathbb{Z}}$-orbits.

  - We start with the boundary of D: Consider the construction of the toroidal boundary divisor D of $\overline{X}_\Sigma^{\mathrm{tor}}$ in the tube domain realization corresponding to the cusp F', just as in section 9.2: A component C of the boundary of D corresponds to a non-isotropic ray $\rho' \in \Sigma(F')$; more precisely, its structure depends on the elements of the star

    $$\mathrm{Star}_{\Sigma(F')}(\sigma_0)$$

    of the two dimensional cone $\sigma_0 \in \Sigma(F)$ in $\Sigma(F')$ spanned by the isotropic ray ρ_0 defining $\overline{F} \supset F'$ and ρ', cf. lemma 9.2.3. Consider the corresponding torus orbits $O(\sigma)$ in the toric variety

    $$((\mathbb{C}^*)^n)_{\Sigma(F)}\,,$$

    then

    $$C = \bigcup_{\sigma \in \mathrm{Star}_{\Sigma(F)}(\sigma_0)} O(\sigma).$$

  - Analogously, a boundary component S of zero-dimensional type of $\overline{X(K_\lambda)}$ over F' is defined by a non-isotropic ray $\rho'' \in \Sigma_{X(K_\lambda)}(F')$ resp. its star

    $$\mathrm{Star}_{\Sigma_{X(K_\lambda)}(F')}(\rho'').$$

Again, we can consider the corresponding torus orbits in

$$\left((\mathbb{C}^*)^{n-1}\right)_{\Sigma_{X(K_\lambda)}(F')} \subseteq ((\mathbb{C}^*)^n)_{\Sigma(F')},$$

then

$$S = \bigcup_{\sigma \in \mathrm{Star}_{\Sigma_{X(K_\lambda)}(F')}(\rho'')} O(\sigma).$$

We come back to the intersection $S \cap D$:

The intersection of the torus orbits of the respective boundary components in $((\mathbb{C}^*)^n)_{\Sigma(F')}$ is non-trivial if and only if the defining cones are identical, so they are of the form $O(\sigma)$ with σ lying in the star of σ_0 (considered in $\Sigma(F')$) and simultaneously in the star of ρ' (considered in $\Sigma_{X(K_\lambda)}(F')$).

Taken together, these conditions imply, as $\Sigma_{X(K_\lambda)}(F') = \Sigma_{X(K_\lambda)}(F') \cap \Sigma(F')$, the condition

$$\rho', \rho'', \rho_0 \preceq \sigma \in \Sigma_{X(K_\lambda)}(F') = \Sigma(F') \cap \lambda^\perp.$$

Since rays in $\Sigma_{X(K_\lambda)}(F')$ are also rays in $\Sigma(F')$, we can restrict our considerations to the case $\rho' = \rho''$ and get

$$S \cap D = \bigcup_{\rho'} \bigcup_{\rho_0, \rho' \preceq \sigma \in \Sigma_{X(K_\lambda)}(F')} O(\sigma),$$

where ρ' runs over a system of non-isotropic rays in a fundamental domain. This is exactly the characterization of the torus orbits in the decomposition of the component of $D_\lambda \subseteq \overline{X(K_\lambda)}$ lying over the cusp $F' \subset \overline{F}$.

In particular, as $\overline{F} = F \cup \bigcup_{F'} F'$, where F' runs over the zero-dimensional cusp as in the second case, and $D_\lambda = D_\lambda^\circ \cup \bigcup_{F' \in \partial \overline{F}} D_{F'}$

$$\overline{X(K_\lambda)} \cap D = D_\lambda$$

and hence $\mathrm{codim}_{\overline{X(K_\lambda)}}\left(\overline{X(K_\lambda)} \cap D\right) = 1$, so the initial assumption is proved as well. Moreover, the preceding proof shows that $\overline{X(K_\lambda)}$ and D intersect transversely (note that $\overline{X(K_\lambda)}$ has λ as a local normal while D has its local normal in K_λ), so

$$\left[D \cap \overline{X(K_\lambda)}\right] = [D_\lambda]$$

without any multiplicities. $\square$

This proof yields a good example of remark 12.2.1: A toroidal boundary component of zero-dimensional type of $\overline{X(K_\lambda)}$ is defined by the $\overline{\mathcal{P}(F,\lambda)}_{\mathbb{Z}}$-orbit of a non-isotropic ray, with $\overline{\mathcal{P}(F,\lambda)}_{\mathbb{Z}}$ denoting the image of $\mathcal{P}(F) \cap \Gamma(K_\lambda)$ in $\mathrm{Aut}(\mathcal{C}(\mathcal{F}) \cap \lambda^\perp)$ as in section 11.2. This acts on the cone in the Siegel domain realization of the special divisor corresponding to λ as in section 6.3. Fixing a fundamental domain and pretending that this divisor is defined by the ray itself (instead of its orbit) makes the argument more tractable.

Remark 12.2.3. Note that this result and its proof make no real use of the actual lattice $II_{2,10}(N)$ but only of the properties of the reflective compactification. In particular: This result is independent of the rank of L, so it works as well for divisors of the reflective compactifications of reflective special cycles (corresponding to lattices $L_i \subseteq L$ of lower rank as in section 11.3).

Nevertheless, one has to be careful in the case of small ranks of the defining lattices L_i: The Baily-Borel closure of the special divisor has no one-dimensional cusps for $\operatorname{rank}(L_i) < 4$, so the statement is vacuously true.
For $\operatorname{rank}(L_i) = 4$ the intersection product consists of divisors on a Hilbert modular surface and therefore yields a linear combination of points whose degree gives an intersection number. For the theory of these Shimura varieties see [HZ76].
As in remark 4.1.7 this implies

$$\left[\overline{X(K_\lambda)}\right] \cdot [D]^l = i_* \left[D_\lambda\right]^l$$

for any $l > 0$.
Lemma 12.2.2 applies only if $F \subseteq \overline{X(L)}^{\mathrm{BB}}$ defines a one-dimensional cusp of $\overline{X(K_\lambda)}^{\mathrm{BB}}$. The next result treats the remaining cases. With the prerequisites of lemma 12.2.2 we have:

Proposition 12.2.4. *If F defines a lower-rank Baily-Borel cusp of $\overline{X(K_\lambda)}^{BB}$ the intersection product*

$$\left[\overline{X(K_\lambda)}\right] \cdot [D]$$

is zero.

Proof. We go back to the proof of lemma 12.2.2: If F defines no cusp of $\overline{X\left(K_\lambda\right)}^{\mathrm{BB}}$, the intersection product as above lies over $\pi_{K_\lambda}(D) \cap \pi_L\left(\overline{X\left(K_\lambda\right)}^{\mathrm{BB}}\right) = \pi_{K_\lambda}(D) = \emptyset$, so it is trivial.
If F defines a zero-dimensional cusp of $\overline{X(K_\lambda)}^{\mathrm{BB}}$ the case of $F' = F$ in the preceding proof is impossible. For the $F' \subsetneq \overline{F}$-case we note the condition

$$\rho', \rho_0 \preceq \sigma \in \Sigma_{X(K_\lambda)}(F') = \Sigma(F') \cap \lambda^\perp$$

on the torus orbits constituting the possible intersection: A necessary condition for this to be non-empty is $\rho_0 \subset \lambda^\perp$ which would be equivalent to having a rank-two isotropic sublattice (spanned by some generator of ρ_0 and a generator of a lattice defining the underlying cusp F') in K_λ. This contradicts the dimension of the cusp of $\overline{X(K_\lambda)}^{\mathrm{BB}}$ defined by F. □

The preceding result is just a technical way of phrasing the following observation:
The intersection of a given one-dimensional Baily-Borel cusp F of $X(L)$ with a special divisor $X(K_\lambda)$ corresponding to $\lambda \in L$ is either trivial (all vectors of the elements of the Γ-orbit of the defining isotropic lattice are not orthogonal to λ, then the cusp F is cut off from $X(K_\lambda)$), zero-dimensional (there is a representative of the Γ-orbit of the

isotropic lattice whose intersection with K_λ has rank 1, so $X(K_\lambda)$ meets $\overline{F}$ only in a zero-dimensional boundary point) or one-dimensional. In the first two cases, the intersection is trivial by either lack of underlying cusps or disjointness of torus orbits.
The last few results enable us to prove the following result. The notation $\mathcal{O}$(toric) just means that any remaining term contains at least one term that can be represented by the cycle class of a smooth compact toric variety.

Corollary 12.2.5. *Suppose that, in the Chow ring of $\overline{X}_\Sigma^{tor}$, we have an equation of the form*

$$[D] = \frac{1}{c_D}\left(\sum_i c_{\lambda_i}\left[\overline{X(K_{\lambda_i})}\right] + \sum_{D^1\neq D} c_{D^1}\left[D^1\right] + \sum_{D^0} c_{D^0}\left[D^0\right]\right),$$

with primitive $\lambda_i \in L$ of negative norm, such that the induced admissible families are strictly compatible with the Coxeter family and D^j running through the toroidal boundary divisors of j-dimensional type of $\overline{X}_\Sigma^{tor}$. Then

$$[D]^2 = [D]\cdot[D] = \sum_i \frac{c_{\lambda_i}}{c_D} i_*\left[\overline{D_{\lambda_i}}\right] + \mathcal{O}(toric).$$

Proof. By proposition 11.3.4, the divisors D^0 are smooth compact toric varieties. We can write

$$\begin{aligned}
[D]\cdot[D] &= \frac{1}{c_D}\left(\sum_i c_{\lambda_i} i_*\left[\overline{X(K_{\lambda_i})}\right] + \sum_{D^1\neq D} c_{D^1}\left[D^1\right] + \sum_{D^0} c_{D^0}\left[D^0\right]\right)\cdot[D] \\
&= \sum_i \frac{c_{\lambda_i}}{c_D}\left[\overline{X(K_{\lambda_i})}\right]\cdot[D] + \mathcal{O}(\text{toric}) \\
&= \sum_i \frac{c_{\lambda_i}}{c_D} i_*\left[D_\lambda\right] + \mathcal{O}(\text{toric}).
\end{aligned}$$

The last equivalence is just the statement of lemma 12.2.2 and we know $\left[\overline{D^1}\right]\cdot[D] = 0$ by lemma 11.3.2. □

We postpone the case of intersection between boundary divisors of one- and zero- dimensional type to the next section to give a unified treatment there.

12.2.2. Divisors of zero-dimensional type

This section treats the cases of intersection products of boundary divisors of zero-dimensional type with divisors of the same type, of one-dimensional type, and with special divisors as in section 11.3. Let $D = D^0$ be a fixed toroidal boundary divisor of zero-dimensional type, lying over a cusp F and corresponding to a non-isotropic ray $\rho \subset U(F)_\mathbb{R}$.
We start our treatment with intersections between compactified special divisors and toroidal boundary divisors of 0-dimensional type. It is largely analogous to the treatment in section 12.2.1.

Intersection with special divisors

As before, fix a toroidal boundary divisor $D \subseteq \overline{X}^{\mathrm{tor}}_{\Sigma}$ of zero-dimensional type corresponding to a non-isotropic ray ρ and a special divisor $X(K_\lambda)$ as in section 11.3, i.e. an embedded Shimura subvariety of $X(L)$ corresponding to a primitive $\lambda \in L$ with negative norm such that the induced admissible family for $X(K_\lambda)$ is strictly compatible with the one for $X(L)$. Denote its induced toroidal compactification by $\overline{X(K_\lambda)}$ with closed immersion

$$i : \overline{X(K_\lambda)} \hookrightarrow \overline{X(L)}.$$

We assume again that F defines a zero-dimensional Baily-Borel cusp of $X(K_{\lambda_i})$, that is, having a representative I' with $I' \subseteq K_\lambda$ of rank one (otherwise the intersection product with $\overline{X(K_{\lambda_i})}$ is trivially zero). Once again we choose a decomposition of L and K_λ in a compatible way as in section 6.3.
We inferred in proposition 11.3.4 that the divisor D is a smooth compact toric variety and determined its defining fan.
The next result will describe the intersection product of $\left[\overline{X(K_\lambda)}\right]$ and $[D]$ in $\mathrm{CH}\left(\overline{X(L)}\right)$ as a toric variety.

Lemma 12.2.6. *If $\overline{\mathcal{P}(F)}_{\mathbb{Z}}\rho$ has a representative in $\mathcal{C}(F) \cap \lambda^{\perp}$ we have set-theoretically*

$$\overline{X(K_\lambda)} \cap D = D_\lambda$$

with $D_\lambda \subseteq \overline{X(K_\lambda)}$ the toric variety corresponding to $\mathrm{Star}_{\Sigma_{\overline{X(K_\lambda)}}(F)}(\rho)$.
Furthermore $\overline{X(K_\lambda)} \cap D = \emptyset$ if $\rho' \nsubseteq \lambda^{\perp}$ for any $\rho' \in \overline{\mathcal{P}(F)}_{\mathbb{Z}}\rho$.

Proof. We mimic the argument in lemma 12.2.2 and distinguish the possible pairwise intersections of strata of $\overline{X(K_\lambda)}$ with D:
First of all, any intersection between D and the interior of $\overline{X(K_\lambda)}$ is trivial.
For a boundary stratum $S \subseteq \overline{X(K_\lambda)}$ over a cusp F' we have

$$\pi_L\left(S \cap D\right) = \pi_L\left(\pi_\lambda^{-1}\left(F'\right) \cap \pi_L^{-1}\left(\overline{F}\right)\right) \subseteq F' \cap \overline{F} = F' \cap F,$$

since $\overline{F} = F = \{pt\}$, so a boundary stratum of $\overline{X(K_\lambda)}$ meets D non-trivially only if $F' = F$. Hence, we can assume that $F' = F$ and the stratum S lies over the same cusp as D. By the characterization of the boundary components of zero-dimensional type in lemma 9.2.3 these correspond to torus orbits $O(\tau)$ for $\tau \in \Sigma_{\overline{X(K_\lambda)}}(F)$. Analogously, D is the union of Γ-orbits of torus orbits $O(\sigma)$ for σ a cone in the star $\mathrm{Star}_{\Sigma(F)}(\rho)$ of a non-isotropic ray $\rho \in \Sigma_F$. By construction of $\Sigma_{\overline{X(K_\lambda)}}(F)$ as the (restriction of the) intersection of $\Sigma(F)$ with $\lambda^{\perp}$ we have

$$\Sigma_{\overline{X(K_\lambda)}}(F) \subseteq \Sigma(F)$$

and

$$\left(\left(\mathbb{C}^*\right)^{n-1}\right)_{\Sigma_{\overline{X(K_\lambda)}}(F)} \subseteq \left(\left(\mathbb{C}^*\right)^{n}\right)_{\Sigma(F)}.$$

We restrict to a fundamental domain of the action of

$$\overline{\mathcal{P}(F)}_{\mathbb{Z}} \cap \mathrm{Aut}(\mathcal{C}(F) \cap \lambda^{\perp})$$

and see that the intersection of D with a boundary component $O(\tau)$ of zero-dimensional type is either equal to $O(\tau)$ (if τ is in the star $\mathrm{Star}_{\Sigma(F)}(\rho)$ of ρ) or empty (if τ is not a cone in the star). More generally, we get

$$\begin{aligned} \overline{X(K_\lambda)} \cap D &= \bigcup_{\tau \in \Sigma_{\overline{X(K_\lambda)}}(F)} O(\tau) \cap D \\ &= \bigcup_{\rho \preceq \tau \in \Sigma(F) \cap \lambda^{\perp}} O(\tau). \end{aligned}$$

If $(\rho, \lambda) = 0$ this is just the description of $X_{\mathrm{Star}_{\Sigma_{\overline{X(K_\lambda)}}}(F)}$ as in lemma 9.2.3; if $(\rho, \lambda) \neq 0$ this union is empty. □

Remark 12.2.7. The intersection is transversal by the same argument as in lemma 12.2.2, so this translates to

$$\left[\overline{X(K_\lambda)}\right] \cdot [D] = \begin{cases} 0 & \text{if } \overline{\mathcal{P}(F)}_{\mathbb{Z}} \rho \cap \lambda^{\perp} = \emptyset \\ i_*[D_\lambda] = i_* \left[X_{\mathrm{Star}_{\Sigma_{\overline{X(K_\lambda)}}(F)}(\rho)}\right] & \text{else} \end{cases}$$

in the Chow ring $\mathrm{CH}\left(\overline{X}_{\Sigma}^{\mathrm{tor}}\right)$ with $i_{\overline{X(K_\lambda)}} : \overline{X(K_\lambda)} \to \overline{X}_{\Sigma}^{\mathrm{tor}}$ denoting as usual the inclusion. The case of empty intersection happens if and only if ρ is in the same Γ-orbit as the weight dual to the root λ. In this case, the restriction of $\overline{X}_{\Sigma}^{\mathrm{tor}}$ to $\lambda^{\perp}$ 'cuts off the direction of the cusp underlying D'.

We can use the triviality of toric logarithmic Chern classes of lemma 10.2.1 and pull everything back via $i^*_{\overline{X(K_\lambda)}}$ to see:

Corollary 12.2.8. *We have*

$$[D_\lambda] \cdot c_i \left(i^*_{\overline{X(K_\lambda)}} \left(\Omega^1_X(\log \Delta) \right) \right) = 0$$

for D_λ as before and Δ the simple normal crossing compactification divisor.

We turn to the previously announced case of intersection products between boundary divisors of distinct type.

Intersection with boundary divisors of 1-dimensional type

Let D^1 be an arbitrary toroidal boundary divisor of one-dimensional type lying over a one-dimensional Baily-Borel cusp F_1. The usual consideration

$$\pi \left(D \cap D^1 \right) \subseteq \pi \left(D \right) \cap \pi \left(D^1 \right) = \overline{F} \cap \overline{F_1} = F \cap \overline{F_1} \subseteq F \cap \partial \overline{F_1}$$

shows that the intersection is empty unless $F \subseteq \overline{F_1}$ lies in the boundary of F_1. In this case, the intersection lies exactly over the zero-dimensional cusp F.

Lemma 12.2.9. *Suppose $\pi\left(D\right) \subseteq \pi\left(D^1\right)$. Then the divisor D^1 corresponds to an isotropic ray $\rho_0 \in \Sigma(F)$ and the intersection $D \cap D^1$ is given by the toric variety $X_{Star_{\Sigma(F)}(\sigma)}$ corresponding to the star $Star_{\Sigma(F)}(\langle\rho_0, \rho\rangle)$ in $\Sigma(F)$.*

Proof. We know from proposition 9.1.4 that the boundary of D^1 is supported on the cusps $F' \subseteq \overline{F_1}$ and we have

$$\begin{aligned}
\overline{D^1} \cap D &= \left(\bigcup_{F' \preceq F_1} \bigcup_{\rho_0 \preceq \sigma \in \Sigma(F')} O(\sigma) \right) \cap \left(\bigcup_{\rho \preceq \sigma' \in \Sigma(F)} O(\sigma') \right) \\
&= \bigcup_{F' \preceq F_1} \bigcup_{\rho_0 \preceq \sigma \in \Sigma(F')} \bigcup_{\rho \preceq \sigma' \in \Sigma(F)} O(\sigma) \cap O(\sigma') \\
&= \bigcup_{\rho_0, \rho \preceq \sigma \in \Sigma(F)} O(\sigma),
\end{aligned}$$

which is exactly the toric variety of the star of $\langle\rho_0, \rho\rangle$ in $\Sigma(F)$ by proposition 2.1.11. □

If ρ_0 and ρ are not contained in any common cone of $\Sigma(F)$, the intersection above is empty as the defining star is empty as well.
Using the codimension and smoothness argument as before, this translates to the following statement on intersection products:

Corollary 12.2.10. *The intersection product $[D] \cdot [D^1]$ as above is zero unless*

$$\pi\left(D\right) \subset \pi\left(D^1\right)$$

and $\langle\rho_0, \rho\rangle \in \Sigma(F)$. In the latter case, we have

$$[D] \cdot \left[D^1\right] = i_* \left[X_{Star_{\Sigma(F)}(\langle\rho_0, \rho\rangle)}\right]$$

with the inclusion morphism $i : D \hookrightarrow \overline{X}_{\Sigma}^{tor}$.

The remaining case involving toroidal boundary divisors of zero-dimensional type are the intersection products of purely zero-dimensional type.

Intersection with boundary divisors of 0-dimensional type

The case of intersection products between toroidal divisors of zero-dimensional type is almost identical to the case just treated, so we simply state the suitably adapted results:

Lemma 12.2.11. *Let $D' \neq D$ be a toroidal boundary divisor of 0-dimensional type over a cusp F' corresponding to a ray ρ'. Then $D \cap D' = \emptyset$ unless $F = F'$ and there are representatives of $\overline{\mathcal{P}(F)_{\mathbb{Z}}\rho}, \overline{\mathcal{P}(F)_{\mathbb{Z}}\rho'}$ commonly spanning a cone $\sigma \in \Sigma(F)$. In that case, the intersection is*

$$D \cap D' = X_{Star_{\Sigma(F)}(\sigma)},$$

the toric variety corresponding to the star $Star_{\Sigma(F)}(\sigma)$. Likewise, the intersection product is

$$[D] \cdot \left[\overline{D'}\right] = i_* \left[X_{Star_{\Sigma(F)}(\langle\rho,\rho'\rangle)}\right],$$

with the inclusion morphism $i : D \hookrightarrow \overline{X}_{\Sigma}^{tor}$, or zero.

The case $D' = D$ can be reduced to intersection products without self-intersection terms by the use of Borcherds relations as before in the as of one-dimensional type.
It remains to consider intersection products of special divisors in more detail.

12.2.3. Special divisors

As always, there are two cases to be treated: self-intersections and non-self-intersections. Again, by using relations in the Chow ring, we will be able to reduce the former to the latter and to intersection products with toroidal boundary divisors. Since the latter case has been treated in the preceding sections of this chapter, we focus here on the intersection products of non-equal special divisors.
The scheme-theoretic intersection of special divisors in orthogonal Shimura varieties has been treated in [Kud97] and [YZZ09]. The setting for this theorem is, of course, more general than our limited treatment here. We restrict their result for general special cycles on Shimura varieties to our situation arising from the $II_{2,10}(N)$-lattice.

Proposition 12.2.12 ([Kud19, Section 4.2])**.** *Let $\lambda_1, \lambda_2 \in L$ of negative norm with $\Gamma\lambda_1 \neq \Gamma\lambda_2$. The scheme-theoretic intersection of the schemes $X(K_{\lambda_1})$ and $X(K_{\lambda_2})$ is*

$$X(K_{\lambda_1}) \cap X(K_{\lambda_2}) = \bigcup_{\gamma \in G} X(\gamma K_{\lambda_1,\lambda_2})$$

with $K_{\lambda_1,\lambda_2} = L \cap \lambda_1^{\perp} \cap \lambda_2^{\perp}$, $G = \Gamma/\Gamma_{\lambda_1} \times \Gamma/\Gamma_{\lambda_2}$ and $\gamma = ([\gamma_1],[\gamma_2])$ acting on K_{λ_1,λ_2} by $\gamma(K_{\lambda_1,\lambda_2}) = K_{\gamma_1\lambda_1,\gamma_2\lambda_2}$.

For sufficiently small $\Gamma = \Gamma(N)$ (for example via large N), the canonical representative $X(K_{\lambda_1,\lambda_2})$, by [YZZ09, Proposition 2.3], is the only one appearing in the union on the right hand side of proposition 12.2.12.
Moreover, one can consider the pullback via $i : X(K_{\lambda_1}) \hookrightarrow X(L)$ of the corresponding cycle classes:

Proposition 12.2.13 ([Kud19, Proposition 6.1])**.** *In the situation as before we have*

$$i^* \left[X(K_{\lambda_2})\right] = \left[X(K_{\lambda_1}) \cap X(K_{\lambda_2})\right] = \sum_{\gamma \in \Gamma_{\lambda_1} \backslash \Gamma / \Gamma_{\lambda_2}} \left[X\left(\left(K_{\lambda_1{}^{\perp}}\right)_{\gamma\lambda_2}\right)\right] \in \mathrm{CH}\left(X\left(K_{\lambda_1}\right)\right).$$

A short remark explaining the notation is adequate: We have $\Gamma_{\lambda_i} = \Gamma(K_{\lambda_i})$ and the divisor

$$X\left(\left(K_{\lambda_1{}^{\perp}}\right)_{\gamma\lambda_2}\right)$$

is nothing but the special divisor corresponding to the image $p(\gamma\lambda_2)$ of $\gamma\lambda_2$ under the orthogonal projection $p : L \to L \cap \lambda_1^{\perp}$; in other words

$$X\left(\left(K_{\lambda_1^{\perp}}\right)_{\gamma\lambda_2}\right) = \pi_{K_{\lambda_1}}\left((\gamma\lambda_2)^{\perp}\right),$$

with $\pi_{K_{\lambda_1}}$ the quotient-by-Γ_{K_1} map and the orthogonal complement taken in $\mathcal{D}_{K_{\lambda_1}}$, this is a special divisor of the orthogonal Shimura variety $X(K_{\lambda_1})$; its image in $\mathrm{CH}\left(X(L)\right)$ is just the image of $i : X\left(K_{\lambda_1,\gamma\lambda_2}\right) \hookrightarrow X(L)$.
Note that the natural morphisms

$$X(\gamma K_{\lambda_1,\lambda_2}) = X(K_{\lambda_1,\gamma\lambda_2}) \to X\left(K_{\lambda_1}\right)$$

and

$$X(\gamma K_{\lambda_1,\lambda_2}) = X(K_{\lambda_1,\gamma\lambda_2}) \to X\left(L\right)$$

are closed immersions for sufficiently small Γ by [Kud19, Proposition 4.13], which is always the case for our choices.
This result can be extended to the intersection product of compactified divisors:

Proposition 12.2.14. *For the closed immersion $i : \overline{X(K_{\lambda_1})} \hookrightarrow \overline{X}_{\Sigma}^{tor}$ we have*

$$i^*\left[\overline{X\left(K_{\lambda_2}\right)}\right] = \sum_{\gamma\in G}\left[\overline{X(\gamma K_{\lambda_1,\lambda_2})}\right] \in \mathrm{CH}\left(\overline{X(K_{\lambda_1})}\right)$$

for $G = \gamma \in \Gamma_{\lambda_1}\backslash\Gamma/\Gamma_{\lambda_2}$ with $\overline{X(\gamma K_{\lambda_1,\lambda_2})}$ considered as subschemes of $\overline{X(K_{\lambda_1})}$ via the closed immersions

$$i_\gamma : \overline{X(\gamma K_{\lambda_1,\lambda_2})} \hookrightarrow \overline{X(K_{\lambda_1})}.$$

Proof. The natural morphisms i_γ are indeed closed immersions since the induced fans for the embedding $X(K_{\lambda_1,\gamma\lambda_2}) \hookrightarrow X(K_{\lambda_1})$ are strictly compatible by proposition 11.3.11. One computes (as set-theoretic intersections on complex points)

$$\begin{aligned} i^*\overline{X\left(K_{\lambda_2}\right)} &= i^*X\left(K_{\lambda_2}\right) \cup \bigcup_{F \text{ cusp of } X(K_{\lambda_2})} i^*D_F \\ &= \bigcup_{\gamma\in G} X(\gamma K_{\lambda_1,\lambda_2}) \cup \bigcup_{\substack{F \text{ cusp of } X(K_{\lambda_2}) \\ D_F \text{ divisor over } F}} i^*D_F \end{aligned}$$

by proposition 12.2.13. The terms in the latter union are computed in lemma 12.2.2 and lemma 12.2.6: We have

$$i^*D_F = \begin{cases} D^1 & F \text{ is a 1-dimensional cusp of } \overline{X\left(K_{\lambda_1,\gamma\lambda_2}\right)} \\ X_{\mathrm{Star}_{\Sigma_{\overline{X(K_{\lambda_1,\gamma\lambda_2})}}}(F)}(\rho) & F \text{ is a 0-dimensional cusp of } \overline{X\left(K_{\lambda_1,\gamma\lambda_2}\right)} \\ \emptyset & F \text{ is not a cusp of } \overline{X\left(K_{\lambda_1,\gamma\lambda_2}\right)} \end{cases}$$

with $D^1 \subseteq \overline{X\left(K_{\lambda_1,\gamma\lambda_2}\right)}$ the toroidal boundary component of 1-dimensional type over F and

$$X_{\mathrm{Star}_{\Sigma_{\overline{X(K_{\lambda_1,\gamma\lambda_2})}}(F)}(\rho)} \subseteq \overline{X\left(K_{\lambda_1,\gamma\lambda_2}\right)}$$

the toric variety of the star of ρ in $\Sigma(F) \cap \lambda_1^{\perp} \cap \gamma\lambda_2^{\perp}$ for D_F corresponding to the orbit $\overline{\mathcal{P}(F)}_{\mathbb{Z}}\rho$. These are the toroidal boundary components occurring in the closures $\overline{X\left(K_{\lambda_1,\gamma\lambda_2}\right)} \subseteq \overline{X}_{\Sigma}^{\mathrm{tor}}$ for $\gamma \in G$, which are themselves a toroidal compactification by virtue of proposition 11.3.11. Taking the union over $\gamma \in G$ moreover yields any of the boundary components occurring in $\overline{X\left(K_{\lambda_1,\gamma\lambda_2}\right)}$ for any $\gamma \in G$ and accounting for possible multiplicities for these boundary components shows that this is in fact the scheme-theoretic intersection whose overall multiplicities for $\overline{X\left(K_{\lambda_1,\gamma\lambda_2}\right)}$ amounts to one, so the intersection is transversal and

$$i^*\left[\overline{X\left(K_{\lambda_2}\right)}\right] = \sum_{\gamma\in G} \overline{X\left(K_{\lambda_1,\gamma\lambda_2}\right)} \in \mathrm{CH}\left(\overline{X(K_{\lambda_1})}\right)$$

as claimed. □

As we stated before, the Coxeter family defines toroidal compactifications for all rescalings $I\!I_{2,10}(N)$. Note that the preceding objects makes sense for any of these rescalings, so we can consider their behavior under changing of the rescaling (equivalently: under changing the level of Γ): For suitably small Γ the preceding sum contains only one representative and one can choose $\gamma = \mathrm{id}$ to get the very pleasing and natural relation

$$\left[\overline{X\left(K_{\lambda_1}\right)} \cap \overline{X\left(K_{\lambda_2}\right)}\right] = \left[\overline{X\left(K_{\lambda_1,\lambda_2}\right)}\right]$$

as cycles in the Chow ring of $\overline{X\left(K_{\lambda_1}\right)}$.

If $\lambda_1 \perp \lambda_2$, the choice of $\Gamma = \widetilde{\mathrm{SO}}^+(I\!I_{2,10}(p))$ for $p \gg 0$ prime is sufficient for this. In general one may need to allow for $\Gamma = \widetilde{\mathrm{SO}}^+(2p) \subseteq \widetilde{\mathrm{O}}^+(p)$, cf. [YZZ09].

We summarize the results of the preceding sections in the following theorem which describes a suitable part of the intersection theory of boundary and special divisors of the reflective compactification $\overline{X}_{\Sigma}^{\mathrm{tor}}$:

Theorem 12.2.15. *Consider the following divisors of* $\overline{X}_{\Sigma}^{tor}$*:*

- *For* $i = 1, 2$ *let* D_i^1 *be distinct toroidal boundary divisors of one-dimensional type over one-dimensional Baily-Borel cusps* $F_{1,1} \neq F_{1,2}$*.*
- *For* $i = 1, 2$ *let* D_i^0 *be distinct toroidal boundary divisors of zero-dimensional type over zero-dimensional Baily-Borel cusps* $F_{0,i}$*, corresponding to the orbits of non-isotropic rays* $\rho_{0,1} \neq \rho_{0,2}$*.*
- *For* $i = 1, 2$ *let* $\lambda_i \in L$ *be primitive vectors with* $\Gamma\lambda_1 \neq \Gamma\lambda_2$ *as in proposition 12.2.12 (i.e. such that the induced admissible families are strictly compatible) and*

  $$\overline{X\left(K_{\lambda_i}\right)} \subseteq \overline{X}_{\Sigma}^{tor}$$

  the corresponding special divisors.

The intersection product $[D_1] \cdot [D_2]$ of divisor classes $[D_1], [D_2]$ is trivial in the following cases (and their symmetric counterparts):

i) $D_1 \neq D_2$ *are toroidal boundary divisors of one-dimensional type*

ii) D_1 *is a toroidal boundary divisor of one-dimensional type over a Baily-Borel cusp* F_1 *and* D_2 *is a toroidal boundary divisor of zero-dimensional type over a Baily-Borel cusp* F_2 *with either*

- $F_2 \not\subset \overline{F_1}$ *or*
- $F_2 \subset \overline{F_1}$*, with* D_1 *being defined inside* $\mathcal{C}(F_1)$ *by the* $\overline{\mathcal{P}(F_2)}_{\mathbb{Z}}$*-orbit of an isotropic ray* ρ_0 *and* D_2 *being defined by* $\overline{\mathcal{P}(F_2)}_{\mathbb{Z}}\rho$ *for a non-isotropic ray* ρ*, and there is no cone* $\sigma \in \Sigma(F_1)$ *with*
$$\rho_0', \rho' \preceq \sigma$$
for any choice ρ_0', ρ' *of orbit representatives.*

iii) $D_1 \neq D_2$ *are toroidal boundary divisors of zero-dimensional type over respective Baily-Borel cusps* F_1, F_2 *with either*

- $F_1 \neq F_2$ *or*
- $F = F_1 = F_2$ *and for* $\rho_1' \in \overline{\mathcal{P}(F_1)}_{\mathbb{Z}}\rho_1$ *and* $\rho_2' \in \overline{\mathcal{P}(F_1)}_{\mathbb{Z}}\rho_2$ *there is no cone* $\sigma \in \Sigma(F)$ *with*
$$\rho_1', \rho_2' \preceq \sigma$$

iv) $D_1 = \overline{X(K_{\lambda_1})} \subseteq \overline{X}_{\Sigma}^{tor}$ *is a special divisor and* D_2 *is a toroidal boundary divisor over a Baily-Borel cusp* F_2 *with*

$$\dim \left(\overline{X(K_{\lambda_1})}^{BB} \cap F_2 \right) < \dim F_2$$

v) $D_1 = \overline{X(K_{\lambda_1})} \subseteq \overline{X}_{\Sigma}^{tor}$ *is a special divisor and* D_2 *is a toroidal boundary divisor of zero-dimensional type over a Baily-Borel cusp* F_2 *with*

$$\dim \left(\overline{X(K_{\lambda_1})}^{BB} \cap F_2 \right) = \dim F_2$$

and $\overline{\mathcal{P}(F_1)}_{\mathbb{Z}}\rho_2 \cap \lambda_1^{\perp} = \emptyset$

In the remaining cases, the intersection theory is given by the push-forward to $\overline{X}_{\Sigma}^{tor}$ *of the entry in the intersection matrix in table 12.1.*
We explain the entries one-by-one:

(1,2): This is the case opposite to the second case of ii) as before, so there are $\overline{\mathcal{P}(F_2)}_{\mathbb{Z}}$*-orbit representatives of* ρ_0' *and* ρ' *spanning a common cone* $\langle \rho_0, \rho_{0,1} \rangle \in \Sigma(F')$*. We assume these to be equal to* $\rho_{0,1}$ *and* ρ_0*, then* $X_{Star_{\Sigma(F)}(\sigma)}$ *is the toric variety of the star of* σ*.*

Table 12.1.: Intersection matrix of divisors on $\overline{X}_{\Sigma}^{\mathrm{tor}}$

.	divisor D_1^1 of one-dimensional type	divisor D_1^0 of zero-dimensional type	special divisor $\overline{X(K_{\lambda_1})}$
divisor D_2^1 of one-dimensional type	0	$X_{\mathrm{Star}_{\Sigma(F_2)}(\langle\rho_0,\rho_{0,1}\rangle)}$	$\overline{K_{\Gamma(N)}^{\mathrm{E}_8\cap\lambda_1^{\perp}}}$
divisor D_2^0 of zero-dimensional type	$*$	$X_{\mathrm{Star}_{\Sigma(F)}(\langle\rho_{0,1},\rho_{0,2}\rangle)}$	$X_{\mathrm{Star}_{\Sigma_{\overline{X(K_{\lambda_1})}}(F_2)}(\rho_{0,2})}$
special divisor $\overline{X(K_{\lambda_2})}$	$*$	$*$	$\sum_{\gamma\in\Gamma/\Gamma_{\lambda_2}} \overline{X(K_{\lambda_1,\gamma\lambda_2})}$

.

(1,3): This is the compactified Kuga-Sato variety $\overline{\mathcal{E}^{(n-3)}}$ of rank $n-3$ over the modular curve $\Gamma(N)\backslash\mathbb{H}$, with the compactification induced from the Coxeter family.

(2,2): This is the case opposite to the second case of iii) as before and analogous to (1,2), so there are $\overline{\mathcal{P}(F)}_{\mathbb{Z}}$-orbit representatives of $\rho_{0,1}$ and $\rho_{0,2}$ spanning a common cone $\sigma\in\Sigma(F)$. We assume $\sigma=\langle\rho_{0,1},\rho_{0,2}\rangle$, then $X_{Star_{\Sigma(F)}(\sigma)}$ is the toric variety of the star of σ.

(2,3): In this case the orbit of $\rho_{0,2}$ has a representative in $\Sigma_{X(K_{\lambda_1})}$ which we assume to be $\rho_{0,2}$ itself, and the intersection product is then represented by the toric variety $Star_{\Sigma_{\overline{X(K_{\lambda})}}(F)}(\rho_{0,2})$ associated to its star in $\Sigma_{X(K_{\lambda_1})}(F_2)$.

(3,3): This is simply a sum of the special divisor defined by $\gamma\lambda_2$.

The push-forward of the first column is induced by the closed immersion $D_1^1\hookrightarrow\overline{X}_{\Sigma}^{tor}$, in the second column by $D_1^0\hookrightarrow\overline{X}_{\Sigma}^{tor}$, and in the third column by $\overline{(K_{\lambda_1})}\hookrightarrow\overline{X}_{\Sigma}^{tor}$.

In particular: Non-vanishing intersection products of toroidal boundary divisors with special divisors can be considered as toroidal boundary divisors of the reflective compactifications of the divisors; intersection products of distinct toroidal boundary divisors are either trivial or can be considered on one of the involved toric varieties.
A short and informal summary of the listed cases of trivial intersection is as follows:
The intersection of boundary divisors is trivial if

- the underlying cusps are too far apart or
- the cusps are sufficiently close, but the defining rays are too far apart in the toric setting (no representatives span a cone of the cone decomposition).

The intersection between a special divisor and a boundary divisor is trivial if the cusp underlying the boundary divisors degenerates via the restriction to the special divisor (that is, becomes lower-dimensional or even trivial).

Remark 12.2.16. Note that most of the results in this section remain true for more general situations: The intersection theory of toroidal boundary divisors and their proof

are mainly independent from the exact form of the chosen defining fan. The notable exception is the claim $[D^1] \cdot [D^2] = 0$ for $D_1 \neq D_2$ of one-dimensional type; this depends on a certain property of Σ (i.e. no cone in any $\Sigma(F)$ contains two isotropic lines as faces) and may fail in more general settings.
The intersection theory concerning special divisors only depends on the applicability of theorem 11.3.8 to see that the closure $\overline{H} \subseteq \overline{X}_{\Sigma}^{\text{tor}}$ of a Heegner divisor can be considered as a toroidal compactification itself. In our case this is a joint property of the special choice of Σ to yield the reflective compactification and H to be a reflective divisor.

We are quite sure that this generalizes well to the case $II_{2,18}$ where an analogous construction of reflective compactification should be possible. For general $\overline{X}_{\Sigma}^{\text{tor}}$ corresponding to more general lattices there may be different choices for the vectors defining special divisors that allow for similar treatment.

12.3. Borcherds products and relations

In the preceding considerations we claimed repeatedly that self-intersection products can be handled by the use of suitable relations in the Chow ring. In this section we will finally prove the existence of these relation by the theory of Borcherds products.
We suppose for this section for the moment that we are considering again a general even lattice L of signature $(2, n)$ instead of our special choice $L = II_{2,10}(N)$.

Borcherds products on toroidal compactifications

We will put to use the theory of Borcherds products as developed in [Bor98] and [Bru02], which allows to explicitly construct rational functions on locally symmetric spaces of orthogonal type with explicitly given divisor and hence allow the use of explicit relations in the Chow ring of $X(L)$.
We recall that any suitable nearly holomorphic vector valued modular forms F yields an orthogonal modular form Ψ_F. If the principal part of F is

$$\sum_{\gamma \in D_L} \sum_{\substack{m \in \mathbb{Z} - q_L(\gamma) \\ m < 0}} c_\gamma(m) q^m \mathbf{e}_\gamma$$

the divisor of Ψ_F is

$$\frac{1}{2} \sum_{\gamma \in D_L} \sum_{\substack{m \in \mathbb{Z} - q_L(\gamma) \\ n < 0}} c_\gamma(m) H(\gamma, m).$$

As Borcherds products are rational functions on $X(L)$, we see now that their divisors induce relations of the form

$$0 = \frac{1}{2} \sum_{\gamma \in D_L} \sum_{\substack{m \in \mathbb{Z} - q_L(\gamma) \\ n < 0}} c_\gamma(m) \left[H(\gamma, m) \right]$$

in the Chow ring $\mathrm{CH}(X(L))$ of $X(L)$.

By results of Bruinier every relation involving Heegner divisors can be realized by the divisor of a suitable Borcherds product if the underlying lattice satisfies certain conditions:

Proposition 12.3.1 ([Bru14]). *Let L be a lattice of signature $(2,n)$, $n > 2$ either of prime level or splitting $II_{1,1} \oplus II_{1,1}(N)$ for some $N \in \mathbb{Z}$. Then: Every relation of cycle classes in* $\mathrm{CH}(X(L))$ *involving only classes of Heegner divisors can be realized as being induced by the Borcherds lift of a nearly holomorphic modular form with respect to the Weil representation of $\overline{L}$ and of weight $1 - n/2$.*

As we are mainly working on toroidal compactifications $\overline{X}_\Sigma^{\mathrm{tor}}$ of $X = X(L)$ it is helpful to have a good description of the extension of these *Borcherds relations* in the Chow ring of $\overline{X}_\Sigma^{\mathrm{tor}}$. Let $\overline{X}_\Sigma^{\mathrm{tor}}$ be a toroidal compactification of $X = X(L)$.

Proposition 12.3.2 ([BZ19, Theorem 5.2]). *Let $n \geq 3$ and $F \in M^!_{1-\frac{n}{2}}(\overline{\rho}_L)$ and Ψ_F as before. Then Ψ_F extends to $\overline{X}_\Sigma^{tor}$ and its divisor there is given by*

$$\begin{aligned} \operatorname{div}(\Psi_F) &= \frac{1}{2}\sum_{\mu\in\Delta_L}\sum_{0>m\in\mu^2/2+\mathbb{Z}} c_\mu(m) H^{tor}(\mu,m) \\ &= \left(\frac{1}{2}\sum_{\mu\in\Delta_L}\sum_{0>m\in\mu^2/2+\mathbb{Z}} c_\mu(-m)\overline{H(\mu,m)}\right) + B(\Psi_F) \end{aligned}$$

where $\overline{H(m,\mu)}$ is the closure in $\overline{X}_\Sigma^{tor}$ and the boundary part $B(\Psi_F)$ of the divisor is

$$\frac{1}{24}\sum_{D^1} CT\Big(E_2\cdot\Big\langle \uparrow_D^L(\Theta_D), F\Big\rangle_L\Big)\cdot D^1 - \sum_{D^0} CT\bigg(\Big\langle\Big\langle \uparrow_{K\oplus\Lambda_N}^{L\oplus\Lambda_N}(\Theta_{K,\omega}), G_N^+\Big\rangle_N, F\Big\rangle_L\bigg)\cdot D^0$$

with D^1 running over the toroidal boundary divisors of one-dimensional type and D^0 over the toroidal boundary divisors of zero-dimensional type.

Here, $\Lambda = \Lambda(D_1)$ as appearing in the coefficient of D^1 is the negative definite lattice obtained by the choice of the two-dimensional isotropic lattice defining the one-dimensional Baily-Borel cusp underlying D^1. Analogously, the lattice $K = K(D^0)$ is the lattice of signature $(1, n-1)$ arising from the isotropic line defining the zero-dimensional cusp underlying D^0 as in section 6.3. The arrow operators $\uparrow$ are defined in proposition 7.1.11 and Θ_D is the theta function of the lattice D as in example 7.1.10. The constant term CT is taken with respect to q. For the exact definition of the remaining objects in the coefficient of the D^0-terms see the original paper [BZ19].
We will shorten the notation for these divisors by denoting it by

$$\operatorname{div}(\Psi_F) = \sum_{m,\mu}\lambda_{m,\mu}\overline{H(m,\mu)} + \sum_{D^1}\lambda_{D^1}D^1 + \sum_{D^0}\lambda_{D^0}D^0.$$

If $\lambda_{D^1} \neq 0$ for some $D^1 = D$, then we have

$$D \sim \frac{1}{\lambda_D}\left(\sum_{m,\mu}\lambda_{m,\mu}\overline{Z(m,\mu)} + \sum_{D^1\neq D}\lambda_{D^1}D^1 + \sum_{D^0}\lambda_{D^0}D^0\right)$$

in the Chow ring of $\overline{X}_{\Sigma}^{\text{tor}}$.
In general, the computation of λ_{D^0} seems to be significantly more difficult than the computation of λ_{D^1}.

Example 12.3.3. We give some examples:

1. Let $L = II_{2,10}$, $\Gamma = \mathrm{SO}^+(L)$ and $F = E_4^2/\Delta = q^{-1} + 504 + \mathcal{O}(q)$, then the divisor of Ψ_F is
$$\overline{H(-1)} + B(\Psi_F)$$
with
$$B(\Psi_F) = \lambda_1 D^1 + \sum_{D^0} \lambda_{D^0} D^0$$
and D^1 the unique boundary divisor of one-dimensional type.

   Here, the D_0 all lie over the unique zero-dimensional Baily-Borel cusp F_0 corresponding to the lattice $II_{1,9}$, and their number is equal to the number of $\mathrm{SO}(II_{1,9})$-orbits of non-isotropic rays in $\mathcal{C}(F_0)$. The coefficient λ_1 is
$$\begin{aligned} \lambda_1 &= \frac{1}{24}\mathrm{CT}\left(E_2 \cdot \left\langle \uparrow_D^L (\Theta_D), F\right\rangle_L\right) \\ &= \frac{1}{24}\mathrm{CT}\left(E_2 \cdot \Theta_{\mathrm{E}_8(-1)} \cdot F\right) \end{aligned}$$
as all the appearing discriminant groups are trivial and we are considering only elliptic modular forms. Using $\mathrm{E}_2 = 1 - 24q - 72q^2 + \ldots$, $\Theta_{\mathrm{E}_8(-1)} = E_4 = 1 + 240q^2 + \ldots$ and $F = q^{-1} + 504 + \ldots$ (cf. chapter 7) this gives
$$\lambda_1 = \frac{1}{24}CT\left[(1 - 24q + \ldots)(1 + \ldots)(q^{-1} + 504 + \ldots)\right] = \frac{1}{24}(504 - 24) = 20.$$
This shows that the orthogonal modular form Ψ_F is a cusp form.

2. Let $L = II_{2,26}$, $\Gamma = \mathrm{SO}^+(L)$ and $F = 1/\Delta = q^{-1} + 24 + \ldots$, then the description of $\mathrm{div}(\Psi_F)$ changes as follows: There are now 24 one-dimensional cusps with corresponding toroidal boundary divisors of one-dimensional type defined by the 23 Niemeier lattices and the Leech lattice. The coefficient $\lambda_1(\Lambda)$ of the boundary divisor of zero-dimensional type corresponding to the Leech lattice Λ is
$$\lambda_1 = \frac{1}{24}CT\left[(1 - 24q + \ldots)(1 + \ldots)(q^{-1} + 24 + \ldots)\right] = \frac{1}{24}(24 - 24) = 0$$
since $\Theta_\Lambda = E_{12} = 1 + 196560q^2 + \mathcal{O}(q^4)$. This shows that the Φ_{12} is not a cusp form.

   This coincides with Kudla's observation in [Kud16]: The vanishing order of Φ_{12} on a toroidal boundary divisor D^1 of one-dimensional type over a cusp F_1 is equal to the Coxeter number h of the lattice $D = J^\perp/J$ for J the isotropic plane defining F_1. The Coxeter number is given by $h = \frac{\#\text{roots of } D}{\mathrm{rank}(D)} = \frac{\#\text{roots of } D}{24}$. As the Leech lattice has no roots, its Coxeter number is zero.

Reflective Borcherds relations

We see that the theory of Borcherds products gives many relations on the Chow ring that enable us to reduce pure self-intersection products to intersections as treated in section 12.2 by replacing some of the factors by rationally equivalent cycles.
One point to note is that, for $L = II_{2,10}(N)$, we were able to describe the intersection of toroidal boundary divisors with special divisors only in the case of the latter being of the form $\Gamma\lambda^\perp$ for $\rho \in L$ of norm $-2N$. The defining and decisive characteristic of these vectors are that the hyperplane defined by it yields an automorphism of L.
This can be turned into a definition for general lattices L of signature $(2, n)$:

Definition 12.3.4. A primitive vector $\rho \in L$ is called *reflective* or a *root* if the reflection σ_ρ in $\rho^\perp$ is an automorphism $\sigma_\rho \in \mathrm{O}^+(L)$. A *reflective divisor* of $\mathcal{D}_L$ resp. $X(L)$ is a divisor of the form $\rho^\perp$ resp. $\Gamma(L)\rho^\perp$ for $\rho \in L$ a root.

As we have seen in section 9.3, neither are general Heegner divisors expected to be irreducible nor are the irreducible components expected to be reflective in general; so a Borcherds product is unlikely to have only reflective divisors in general. The modular forms satisfying this conditions are called *reflective modular forms*:

Definition 12.3.5. A holomorphic orthogonal modular form on $\Gamma\backslash\mathcal{D}_L$ is called *reflective* if its divisor is contained in the union

$$\bigcup_{\rho\in L \text{ root}} \Gamma\rho^\perp.$$

It is said to be *strongly reflective* if the vanishing order at an irreducible component of its divisor is at most 1. If the divisor is equal to the union above, the modular form is said to be *completely reflective*. If one restricts in the union to vectors of length -2, the corresponding notion is called 2-reflectiveness. A divisor of this form is called a *reflective divisor*.

Fortunately, the construction and classification of reflective modular forms has been the focus of a wide range of research, cf. [Sch06, Ma17, Wan18, Dit19].
Under some assumptions on the lattice L, the reflectiveness of a Borcherds Product Ψ_F can be characterized by properties of the weakly holomorphic modular form F:

Proposition 12.3.6 ([Sch06, Section 9])**.** *Assume L to have square-free odd level with discriminant form Δ_L. The Borcherds product Ψ_F is reflective if the weakly holomorphic modular form F of weight $1 - n/2$ satisfies:*

- *If $\gamma \in \Delta_L$ has order dividing m and norm $1/m$, the Fourier expansion of the component function F_γ is*

$$F_\gamma = c_\gamma(-1/m)q^{-1/m} + \mathcal{O}(1).$$

- *$F_\gamma = \mathcal{O}(1)$ for all other $\gamma \in \Delta_L$.*

There is a converse theorem for this if L splits a hyperbolic plane.
This criterion shows that the Borcherds product corresponding to $E_4^2/\Delta = q^{-1}+504+\ldots$ as in example 12.3.3 is reflective.
Note that reflectivity of an orthogonal modular form is a property that can be checked naturally on the symmetric domain $\mathcal{D}_L$ instead of the locally symmetric space $\Gamma\backslash\mathcal{D}_L$, so it is independent of the actual choice of Γ:

Lemma 12.3.7. *If F is a reflective modular form on $\mathcal{D}_L$ with respect to $\widetilde{\mathrm{SO}}^+(L)$, it is also reflective with respect to any finite-index $\Gamma \subseteq \widetilde{\mathrm{SO}}^+(L)$.*

In particular this applies to the following scenario: Let L be an even lattice of signature $(2,n)$ and let $N > 0$ be an integer. A reflective modular form on L with respect to $\widetilde{\mathrm{SO}}^+(L)$ induces a reflective modular form on any scaling $L(N)$ with respect to $\widetilde{\mathrm{SO}}^+(L(N))$. This follows by noting that the latter group is a finite-index subgroup of $\widetilde{\mathrm{SO}}^+(L)$ and the reflective vectors in $L(N)$ correspond bijectively to the reflective vectors in L: Note that the space $\rho^\perp$ is invariant under scaling of ρ and $\mathrm{O}(L) \cong \mathrm{O}(L(N))$, so, after considering $L(N)$ as $\sqrt{N}L \subset L\otimes\mathbb{R}$, a $\rho \in L$ is reflective if and only if $\sqrt{N}\rho \in L(N)$ is reflective.
This gives us a reflective modular form on $L = II_{2,10}(N)$ for any odd N by applying this construction to $\Psi_{E_4^2/\Delta}$:

Example 12.3.8. The Borcherds product corresponding to E_4^2/Δ as in example 12.3.3 on $II_{2,10}$ is (strongly) reflective by the criterion in proposition 12.3.6. Moreover, as any reflective vector of $II_{2,10}$ has norm -2 (cf. [Sch06, Proposition 2.5]), this form is completely reflective with divisor $H(-1)$. The preceding considerations show that this form can be considered as a reflective orthogonal modular form on $II_{2,10}(N)$. The divisor of this form is given by $H(0,-1)$. Decomposing this Heegner divisor into irreducible components we get exactly the divisors identified in section 11.3 as being compatible with the reflective toroidal compactification on $II_{2,10}(N)$.

We want to prevent a common point of confusion:
Note that this is not the same as the modular form obtained by the map $\uparrow_{II_{2,10}}^{II_{2,10}(N)}$ as in section 7.2. The latter makes use of the inclusion $II_{2,10}(N) \subset II_{2,10}$ and the corresponding natural identification of the respective symmetric spaces. This differs from the identification of the symmetric spaces used in the preceding sections.

Remark 12.3.9. In general, reflective modular forms seem to be rare. There are some classification results ([Sch17]) for reflective modular forms with additional properties like being of singular weight (or some splitting condition).
Moreover, there are some finiteness results about reflective lattices [Ma17]; these are lattices that support a reflective form: Up to scaling and isomorphy there are only finitely many of them and their genera can be explicitly stated.
Unfortunately, this theory gives no information about the actual number of reflective modular forms on the lattices. In particular, we know from the preceding example that

the lattices $II_{2,10}(N)$ are reflective, but there may be several reflective modular forms on them, as the classification result mentioned before do not apply for lack of the splitting of an unscaled hyperbolic lattice.

A good supply of reflective modular forms is important for the intersection theory on the reflective compactification: As we stated in the beginning of this section, self-intersection products can be translated to actual intersection of properly intersecting divisors. This is illustrated in corollary 12.2.5 for the case of a self-intersection of rank 2.
We generalize these notions and constructions to higher special cycles.

Reduction to special divisors and special cycles

Let now again $L = II_{2,10}(N)$ and let $\overline{X}^{\text{tor}}_{\Sigma}$ be the reflective compactification.
Since we are not only interested in the intersection theory of the reflective compactification itself but also in the theory of intersection of divisors on reflective special divisors, it is helpful to be able to apply the intersection theory developed above inductively.
For a reflective divisor $\overline{X(K_\lambda)}$ the results about the intersection theory of toroidal boundary divisors with each other remain true as we saw in remark 12.2.16, at least if the divisors are pairwise distinct. Again, the problem of the resolution of self-intersections remains: To resolve this in the same manner as in the last section, we need to use relations on $\overline{X(K_\lambda)}$, every appearing divisor of which is either

- a toroidal boundary divisor of $\overline{X(K_\lambda)}$ or
- a special divisor D on $\overline{X(K_\lambda)}$ such that D is the toroidal compactification of a special divisor $D \subset X(K_\lambda)$ induced by the fan defining the toroidal compactification of $X(K_\lambda)$. In other words: The induced admissible family on D as in proposition 3.2.2 should be strictly compatible to the admissible family on $X(K_\lambda)$.

As the induced toroidal compactification on $X(K_\lambda) \subseteq \overline{X}^{\text{tor}}_{\Sigma}$ is again called reflective, we call such special divisors *reflectively compatible divisors*. This definition obviously generalizes to cycles of higher codimension.
We note that reflectively compatible divisors need not be reflective for the lattice K_λ in the sense of definition 12.3.5: The compatibility is a condition with respect to the fan Σ which is constructed by roots of $II_{2,10}$; these do not necessarily coincide with roots of K_λ.
Recalling proposition 11.3.11 we get the following reformulation in this new terms:

Proposition 12.3.10. *Define $L_0 = II_{2,10}(N)$ and $L_i = L_{i-1} \cap \lambda_i^{\perp}$ inductively by choosing a root $\lambda_i \in L_{i-1} \subseteq L$ of L, then*

$$\overline{X(L_i)} \subseteq \overline{X(L_{i-1})} \subseteq \overline{X(L_0)} = \overline{X}^{tor}_{\Sigma}$$

via the canonical closed immersions. For any $1 \leq i \leq m-1$ the cycle $\overline{X(L_i)}$ is a reflectively compatible cycle of $\overline{X}^{tor}_{\Sigma}$.

The theory of quasi-pullback as in section 7.2 gives us a powerful tool for the explicit generation of Borcherds relations on $\overline{X(L_i)}$ whose non-boundary constituents are reflectively compatible cycles on $\overline{X}_{\Sigma}^{\mathrm{tor}}$: by quasi-pulling back of reflective orthogonal modular forms on L_0 to L_i.
Even better, we can describe the coefficients of the appearing cycles effectively in terms of the reflective input function and the lattice $\langle\lambda_1, \dots, \lambda_i\rangle$:

Proposition 12.3.11. *Let $\lambda_1, \dots, \lambda_{m-1} \in L = II_{2,10}(N)$ with the corresponding L_i as in the last proposition and $\Psi = \Psi_F$ a reflective orthogonal modular form realized as the Borcherds lift of a vector valued modular form F. Furthermore, let $K_i = \langle\lambda_1, \dots, \lambda_i\rangle$ the negative definite lattice spanned by the given roots and Θ_{K_i} its theta series. The quasi-pullback*

$$\Psi\|_{\mathcal{D}_{L_i}} = \Psi\left(\langle\uparrow_L^{L_i\oplus K_i} F, \Theta_{K_i}\rangle\right)$$

is an orthogonal modular form. Considered as a rational function on $X(L_i)$ it extends to the toroidal compactification $\overline{X(L_i)}$. Its divisor there consists of toroidal boundary divisors and compactified Heegner divisors, the irreducible components of the latter being reflectively compatible divisors.

Proof. This is mainly a collection of results already stated: The modularity of the quasi-pullback and its characterization as a Borcherds lift is in proposition 7.2.5 and theorem 7.2.6, the description of the divisor of a modular form on toroidal compactifications is proposition 12.3.2. It remains to justify the claim about the reflective compatibility: This can be checked on $\mathcal{D}_{L_i}$ and is equivalent to the claim that the divisor of $\Psi\|_{\mathcal{D}_{L_i}}$ is supported only on roots of L. By [Ma19, Proposition 3.4] the orthogonal modular forms

$$\Psi\|_{\mathcal{D}_{L_i}} = \Psi\left(\langle\uparrow_L^{L_i\oplus K_i} F, \Theta_{K_i}\rangle\right)$$

and

$$\Psi\left(\langle F, \Theta_{K_i}\rangle\right)$$

have the same divisor, so we can work with the latter instead. By [Ma19, Proposition 3.5] its divisor is supported on $\lambda^\perp \cap \mathcal{D}_{L_i}$ for $\lambda \in L'$ with $\lambda \notin K'$ and $\lambda^\perp \cap \mathcal{D}_{L_i} \neq \emptyset$ such that the Fourier coefficient $c(F, \lambda + L, q(\lambda))$ is non-zero. As the original $\Psi(F)$ is assumed to be reflective, this shows that $\lambda^\perp = \rho^\perp$ for a root ρ of L. $\square$

Since we proved the existence of the completely reflective modular form

$$\Psi\left(\frac{E_4^2}{\Delta}\right)$$

on $II_{2,10}$ by example 12.3.8, this gives a Borcherds relation with reflectively compatible special divisors on $\overline{X(L_i)}$ for any choice of lattices $L_i \subseteq II_{2,10}(N)$ that are cut out by roots as before.

Remark 12.3.12. There is also a theory of pulling back reflective Borcherds products on a lattice L to actual reflective Borcherds products on a sublattice $K \subset L$, see [Gra09].

To summarize our findings in this chapter: We described the intersection theories on the toroidal boundary divisors as well as the intersection theory of these divisors with each other and with/of reflective special divisors; moreover, we generated the relations in the Chow ring needed to reduce pure self-intersection products to the type of intersection products defined before. Finally, we generalized the construction of these relations to higher special cycles.
This allows us to return to our overall task of determining dimension formulas for orthogonal cusp forms. The next two chapters will use this intersection theory to give better description of the coefficients of the error term in the dimension formula.

13. The linear term

We want to apply the general theory of dimension formulas for orthogonal modular forms in section 10.1 to the $II_{2,10}(N)$-lattice with neat discriminant kernel and the reflective compactification $\overline{X} = \overline{X}_\Sigma^{\mathrm{tor}}$.

We determined the dimension of the space of cusp forms of geometric weight k for a neat discriminant kernel $\Gamma \subseteq \mathrm{O}^+(II_{2,10}(N))$ as a polynomial of k up to a linear polynomial $E(k)$ and described the latter in proposition 10.2.5 as

$$\sum_{l=1}^{n-1} \lambda_{(l)} \sum_{\substack{D \text{ of one-} \\ \text{dimensional type}}} [D^l] Q_{n-l}\left(c_1\left(\Omega^1_{\overline{X}}(\log\Delta)^{\otimes(1-k)}\right); \mathrm{c}\left(\Omega^1_{\overline{X}}(\log\Delta)\right)\right) + \mathcal{O}(1)$$

with $\lambda_{(l)} = \frac{(-1)^{l+1}}{2^l}$ from table 4.1. Here Q_{n-l} is the universal polynomial of dimension $n-l$ as in corollary 4.1.18 with

$$\mathrm{c}(\Omega^1_{\overline{X}}(\log\Delta)) = \left(c_1\left(\Omega^1_{\overline{X}}(\log\Delta)\right), \ldots, c_n\left(\Omega^1_{\overline{X}}(\log)\right)\right),$$

the total Chern class of the logarithmic cotangent bundle with respect to the boundary divisor $\Delta = \overline{X}_\Sigma^{\mathrm{tor}} \setminus X(L)$ as in definition 4.1.22.

In this chapter we will use the intersection theory developed in the preceding chapter and describe this error term even more closely. The first section will reduce these intersection products to logarithmic Euler characteristics; the second section will reduce these further to actual Euler characteristics of bundles of modular forms on the usual modular curve. The strategy we follow here originates in (the slides of) a talk[1] given by Andrew Fiori at the Algebraic Geometry Seminar at Queen's University, where he sketches a possible approach to the computation of the error term. Also, the main tools in the first part of this chapter are from section 4.3, which summarizes the results of Fiori's more general work in [Fio17].

13.1. Reduction via Borcherds relations

We localize the preceding expression

$$\sum_{l=1}^{n-1} \lambda_{(l)} \sum_{\substack{D \text{ of one-} \\ \text{dimensional type}}} [D^l] Q_{n-l}\left(c_1\left(\Omega^1_{\overline{X}}(\log\Delta\right)^{\otimes(1-k)}); \mathrm{c}\left(\Omega^1_X(\log\Delta)\right)\right) + \mathcal{O}(1)$$

[1] Currently available at `https://www.cs.uleth.ca/~fiori/miscwriting.html`, titled "Dimension of Spaces of Modular Forms"

by the use of the uniformity of the reflective compactification:
The locally symmetric space $X(I\!I_{2,10})$ has a unique one-dimensional Baily-Borel cusp and $X(I\!I_{2,10}(N))$ is a covering of it, so for any two one-dimensional Baily-Borel cusps F, F' of $X(I\!I_{2,10}(N))$ there is an automorphism of $X(I\!I_{2,10}(N))$ carrying one to the other. Going back to the construction of the Coxeter family, we see that the partial compactifications over both these cusps, and hence the toroidal boundary divisors, are isomorphic as well, so the local geometries and the appearing intersection numbers coincide.
This shows that we can reduce to a fixed toroidal boundary divisor of one-dimensional type: We have

$$E(k) = \nu_1 \sum_{l=1}^{n-1} \lambda_{(l)} [D^l] Q_{n-l} \left(c_1 \left(\Omega^1_{\overline{X}}(\log \Delta)^{\otimes(1-k)} \right); \mathrm{c}\left(\Omega^1_{X}(\log) \right) \right) + \mathcal{O}(1)$$

with a fixed toroidal boundary divisor D of one-dimensional type and ν_1 the number of one-dimensional Baily-Borel cusps of $X(I\!I_{2,10}(N))$.
To compute the linear coefficient of the error term, it suffices to treat the constituents of the form $[D]^l Q_{n-l}(\ldots)$.
To shorten notation, we define for $1 \leq l \leq n-1$ the intersection number

$$I_{D,l,\overline{X}} = [D]^l Q_{n-l} \left((1-k) c_1 \left(\Omega^1_{\overline{X}}(\log \Delta) \right), \mathrm{c}\left(\Omega^1_{\overline{X}}(\log \Delta) \right) \right) \in \mathrm{CH}^n\left(\overline{X} \right).$$

Remark 13.1.1. The computation of this number is easy for $l = 1$: Note that we get

$$\begin{aligned} & [D] Q_{n-1} \left((1-k) c_1 \left(\Omega^1_{\overline{X}}(\log \Delta) \right), \mathrm{c}\left(\Omega^1_{\overline{X}}(\log \Delta) \right) \right) \\ = \; & i_D^* Q_{n-1} \left(c_1 \left(\Omega^1_{\overline{X}}(\log \Delta)|_D^{\otimes(1-k)} \right), \mathrm{c}\left(\Omega^1_{\overline{X}}(\log \Delta \cap D) \right) \right) \\ = \; & \chi\left(D, \Delta \cap D, \Omega^n_{\overline{X}}(\log \Delta)|_D^{\otimes(1-k)} \right) \end{aligned}$$

by proposition 4.3.4, so this an actual logarithmic Euler characteristic on the divisor D with $\Delta \cap D$ the boundary divisor of the corresponding closure of the toroidal boundary component inside the toroidal compactification $\overline{X}^{\mathrm{tor}}_{\Sigma}$.
By proposition 9.1.4 we know that this component is the Kuga-Sato variety of rank $n-2$ over the modular curve $Y(\Gamma_1) = \Gamma_1 \backslash \mathbb{H}$ which is the underlying Baily-Borel cusp.

We denote the Kuga-Sato variety compactified in this way as before by $\overline{\mathcal{E}^{(n-2)}}$. This logarithmic Euler characteristic carries a lot more structure than the original intersection product and will be easier to handle. We will generalize this to $I_{D,l,\overline{X}}$ for $l \geq 2$: this will be done via the usage of reflectively compatible Borcherds relations as introduced in the last chapter.
To clarify this strategy we will start with a simplified approach with a strong conjectural assumption.

Conjectural Digression

This section is somewhat conjectural as it assumes the existence of certain Borcherds relations which we are not able to construct at the moment. The next section will give a complete (albeit more complicated) treatment of the arguments here while relying solely on the existence of relations we constructed in section 12.3.
We start with the following lemma that decomposes intersection products related to the constituents of the linear term:

Lemma 13.1.2. *Let $0 \leq l-1, m \leq n-1$ with $2 \leq l+m \leq n$. Let D be a fixed boundary divisor of $\overline{X}^{tor}_{\Sigma}$ of one-dimensional type. Suppose that there is a Borcherds relation of the form*

$$D \sim \frac{1}{\lambda_D}\left(\lambda_E E + \sum_{D^1 \neq D} \lambda_{D^1} D^1 + \sum_{D^0} \lambda_{D^0} D^0\right),$$

with $E = \overline{X(K_\lambda)}$ a reflective divisor. Then: The intersection number

$$I = [E]^m [D]^l Q_{n-(l+m)}\left((1-k)c_1\left(\Omega^1_{\overline{X}}(\log \Delta)\right), \mathrm{c}\left(\Omega^1_{\overline{X}}(\log \Delta)\right)\right) \in \mathrm{CH}^n\left(\overline{X}\right)$$

for $\overline{X} = \overline{X}^{tor}_{\Sigma}$ can be written as $I = R + P$ with

i)

$$R = \sum_{i=1}^{(n-1)-1} c_i I_{D_\lambda, i, E},$$

where $I_{D_\lambda,i,E}$ is defined to be

$$[D_\lambda]^i Q_{n-i-1}\left((1-k)c_1\left(i_E^*\left(\Omega^1_X(\log \Delta)\right)\right), \mathrm{c}\left(\Omega^1_E(\log \Delta \cap E)\right)\right) \in \mathrm{CH}_0\left(E\right)$$

with D_λ being the boundary divisor of one-dimensional type of $E = \overline{X(K_\lambda)}$ lying over the Baily-Borel cusp underlying D and the constants $c_i \in \mathbb{Q}$ depend only on the coefficients in the relation and the parameters l, m.

ii)

$$P = \sum_{l+m=n} c_{m,l} [E]^m [D]^l$$

is a sum of pure intersection products and the coefficients depend again only on the relation.

Proof. If $n = l + m$, there is nothing to show as the intersection product in question does not contain any logarithmic Chern classes and only the terms of case *ii*) appear, so we can assume $2 \leq l + m < n$.
We remember from lemma 10.2.1 and lemma 11.3.2 that

$$[D_1] \cdot [D] = 0 = [D^0] \cdot c_\alpha\left(\Omega^1_X(\log \Delta)\right),$$

so

$$
\begin{aligned}
[E]^m[D]^l Q_{n-(l+m)}(\dots) &= \left[\frac{\lambda_D}{\lambda_E}D\right]^m [D]^{l-1}\left[\frac{\lambda_E}{\lambda_D}E\right] Q_{n-(l+m)}(\dots) \\
&= \left(\frac{\lambda_D}{\lambda_E}\right)^{m-1}[D]^{l+m-1}[E]Q_{n-(l+m)}(\dots).
\end{aligned}
$$

Applying lemma 4.3.8 and corollary 12.2.8 we get

$$
\begin{aligned}
&= \left(\frac{\lambda_D}{\lambda_E}\right)^{m-1}(i_E)_*\left[i_E^*\left([D]^{l+m-1}\right)\cdot Q_{n-(l+m)}\left(i_E^*(\Omega^1_X(\log\Delta)); \mathrm{c}\left(\Omega^1_E(\log\Delta\cap E)\right)\right)\right] \\
&\quad -\left(\frac{\lambda_D}{\lambda_E}\right)^{m-1}\sum_{k=1}^{n-(l+m)}\delta_{(k)}[D]^{l+m-1}[E]^{k+1}Q_{n-(l+m)-k}\left(\Omega^1_X(\log\Delta); \mathrm{c}\left(\Omega^1_X(\log\Delta)\right)\right) \\
&= \left(\frac{\lambda_D}{\lambda_E}\right)^{m-1}(i_E)_*\left([D_\lambda]^{l+m-1}\cdot Q_{n-(l+m)}\left(i_E^*(\Omega^1_X(\log\Delta)); \mathrm{c}\left(\Omega^1_E(\log\Delta\cap E)\right)\right)\right) \\
&\quad -\left(\frac{\lambda_D}{\lambda_E}\right)^{m-1}\sum_{k=1}^{n-(l+m)}\delta_{(k)}[D]^{l+m-1}[E]^{k+1}Q_{n-(l+m)-k}\left(\Omega^1_X(\log\Delta); \mathrm{c}\left(\Omega^1_X(\log\Delta)\right)\right).
\end{aligned}
$$

As proper push-forwards are irrelevant for the computation of intersection numbers, the first term is just a multiple of $I_{D_\lambda,l+m-1,E}$. It remains to further examine the latter sum but this turns out to be recursive:
The term for $n-(l+m)=k$ corresponds to case *ii*) and the remaining ones are of the form of the expression we started with; moreover

$$
n \geq l+m+k = (l+m-1)+(k+1) = l+m+1 > l+m \geq 1
$$

for $n-(l+m) \geq k \geq 1$, so we can apply this procedure again to these terms. Since the index of the Q-polynomial drops by at least one, this procedure stops eventually once the procedure yields only terms as in case *ii*). The claim about the nature of the coefficients obviously follows from all this. $\square$

We come back to the computation of

$$
I_{D,l,X} = [D]^l Q_{n-l}\left((1-k)c_1\left(\Omega^1_{\overline{X}}(\log\Delta)\right), \mathrm{ch}\left(\Omega^1_{\overline{X}}(\log\Delta)\right)\right).
$$

The preceding lemma allows us to identify another logarithmic Euler characteristic on a Kuga-Sato variety:
The number $I_{D,l,X}$ is the special case $m=0$ of the preceding lemma, so it can be expressed as pure intersection numbers and terms of the form $I_{D_\lambda,i,E}$. By a computation completely analogous to the one for $I_{D,1,X}$ we see that

$$
\begin{aligned}
I_{D_\lambda,1,E} &= [D_\lambda]^i Q_{n-i-1}\left((1-k)c_1\left(i_E^*\left(\Omega^1_X(\log\Delta)\right)\right), \mathrm{ch}\left(\Omega^1_E(\log\Delta\cap E)\right)\right) \\
&= \chi\left(D_\lambda, \Delta'', \Omega^n_{\overline{X}}(\log\Delta)|_{D_\lambda}^{\otimes(1-k)}\right)
\end{aligned}
$$

with Δ'' the boundary divisor of the divisor D_λ via the compactification by closure in $\overline{X(K_\lambda)} = E \subseteq \overline{X}^{\text{tor}}_\Sigma$. We note that D_λ is again a compactification of the Kuga-Sato variety over $\Gamma_1\backslash\mathbb{H}$, this time of rank $(n-2)-1=(n-3)$, so we denote it by $\overline{\mathcal{E}^{(n-3)}}$.

Remark 13.1.3. Note that the pure intersection terms do not depend on k, so they do not contribute to the linear term of k.

We observe that the remaining terms $I_{D_\lambda,i,E}$ have the same structure as $I_{D,l,X}$, i.e. one obtains the former from the latter by the substitutions $D \to D_\lambda$, $X \to E$. Hence, the computation of this number amounts to the same type of considerations on the lower-dimensional locally symmetric space E.
This opens up the possibility for another reductive approach by applying this reasoning with new reflectively compatible Borcherds relation on E.

Proposition 13.1.4. *Fix a toroidal boundary divisor $D_0 := D \subseteq \overline{X}^{tor}_\Sigma =: E_0$ of one-dimensional type. Given a chain of lattices $\ldots \subseteq L_{i+1} \subseteq L_i \subseteq \ldots, \subseteq L = L_0$ as in proposition 11.3.11 with the corresponding reflective compactifications $\overline{X(L_i)} \subseteq \overline{X}^{tor}_\Sigma$ of the embedded Shimura subvarieties. Suppose that for any $i = 0, \ldots, n-1$ there is a reflectively compatible Borcherds relation*

$$D_i \sim \frac{1}{\lambda_{D_i}} \left(\lambda_{E_i} E_i + \sum_{D_i^1 \neq D} \lambda_{D_i^1} D_i^1 + \sum_{D_i^0} \lambda_{D_i^0} D_i^0 \right)$$

of cycles on $\overline{X(L_i)}$. Here D_i is the toroidal boundary divisor of one-dimensional type of the embedded Shimura variety $\overline{X(L_i)}$ with $\psi(D_i) \subseteq D$ under the closed immersion $\overline{X(L_i)}^{BB} \hookrightarrow \overline{X(L)}^{BB}$ (cf. section 11.3), $E_i = \overline{X(L_{i+1})}$ is the reflective special divisor defined by λ_i and D_i^0, D_i^1 run through the remaining toroidal boundary divisors of zero- resp. one-dimensional type of $\overline{X(L_i)}$. Then: For any $1 \leq l \leq n-1$ the intersection product

$$I_{D,l,X} = [D]^l Q_{n-l}\left((1-k)c_1\left(\Omega^1_{\overline{X}}(\log\Delta)\right), \mathrm{ch}\left(\Omega^1_{\overline{X}}(\log\Delta)\right)\right) \in \mathrm{CH}^n(X)$$

can be written as

$$I_{D,l,X} = R + P$$

with

i)

$$R = \sum_{i=0}^{n-1} c_i I_{D_i,1,E_i} = \sum_{i=0}^{n-1} c_i \chi\left(D_i, \Delta^{(i)}, \Omega^n_{\overline{X}}(\log\Delta)|_{D_i}^{\otimes(1-k)}\right)$$

with $\Delta^{(i)}$ the boundary divisor of the compactification D_i of the toroidal boundary component of one-dimensional type of E_i corresponding to $D_0 \subseteq \overline{X}^{tor}_\Sigma$. The constants $c_i \in \mathbb{Q}$ depend only on the coefficients in the relations and the dimension, but not on k or Σ.

ii)

$$P = \sum_{i=0}^{n-1} \sum_{l+m=n-i} c_{i,m,l}[E_i]^m[D_i]^l$$

is a sum of pure intersection products and the coefficients depend again only on the relations.

Proof. This is just a repeated application of lemma 13.1.2 with new relations on each embedded Shimura subvariety: We note corollary 12.2.8, so we can exactly replicate the arguments of lemma 13.1.2, even if we changed the sheaf in question from $\Omega^1_{\overline{X}^{\text{tor}}_\Sigma}$ to its pullback $i^*_{E_i}\left(\Omega^1_{\overline{X}^{\text{tor}}_\Sigma}\right) = \Omega^1_{\overline{X}^{\text{tor}}_\Sigma}|_{E_i}$. Hence, we can rewrite any I_{D_i,l,E_i} as a sum of

- pure intersection products as appearing in R and
- $I_{D_{i+1,l,E_{i+1}}}$ with l running from 1 to $\dim E_i - 1$.

By splitting off the terms as in the first case and the term for $l = 1$ in the second case and reapplying lemma 13.1.2 on E_i, we can reduce to the case of $\dim E_{n-2} = 2$ where only pure intersection terms and $I_{D_{n-1},1,E_{n-1}}$ remain. We finally get

$$I_{D_i,1,E_i} = \chi\left(D_i, \Delta^{(i)}, \Omega^n_{\overline{X}}(\log\Delta)|^{\otimes(1-k)}_{D_i}\right),$$

by the same arguments as before, so everything is proved. ☐

We emphasize that the coefficients $c_i, c_{i,m,l}$ do not need to differ from zero. Apart from simple numerical cancellation this can also happen if one of the intermediate intersection products $[E_i]\cdot[D_i]$ happens to vanish by any of the reasons described in theorem 12.2.15. We note again that the pure intersection terms do not depend on k. Moreover the characterization of the D_i as toroidal boundary divisors of one-dimensional type in proposition 9.1.4 still applies, so D_i is the closure of a toroidal boundary divisor of one-dimensional type, which is again the Kuga-Sato variety of rank $n-2-i$ over the modular curve $Y(\Gamma_1) = \Gamma_1\backslash\mathbb{H}$. We write $D_i = \overline{\mathcal{E}^{(n-2-i)}}$ in accordance to our previous notation.
We summarize the consequences of this result for the linear term of $E(k)$:

Corollary 13.1.5. *Given the existence of reflectively compatible Borcherds relations as in proposition 13.1.4, the linear term of $E(k)$ is*

$$\nu_1 \sum_{l>0} \frac{(-1)^{l+1}}{2^l} c_i \chi\left(\overline{\mathcal{E}^{(l)}}, \Delta^{(i)}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}}\right) + \mathcal{O}(1)$$

with ν_1 the number of one-dimensional cusps of $\overline{X}^{BB}$ and $c_i \in \mathbb{Q}$ not depending on k.

Note that our notation $\overline{\mathcal{E}^{(l)}}$ for the Kuga-Sato varieties suppresses the dependence on the position of E_i inside $\overline{X}^{\text{tor}}_\Sigma$ which determines the exact form of the compactification of $\mathcal{E}^{(l)}$.

Unconditional reduction via known Borcherds relations

Unfortunately, the assumption on the existence of reflectively compatible Borcherds relations of the form needed in proposition 13.1.4 is stronger than we can prove at the moment. In general, a reflectively compatible Borcherds relation will be of the form

$$D \sim \frac{1}{\lambda_D}\left(\sum_{E} \lambda_E E + \sum_{D^1 \neq D} \lambda_{D^1} D^1 + \sum_{D^0} \lambda_{D^0} D^0\right)$$

where the first sum runs over a potentially huge number of reflective special divisors. Note that the assumption in proposition 13.1.4 may be even stronger than the existence of a Borcherds relation with only one reflective Heegner divisor if the appearing Heegner divisor is not irreducible.
This has a serious consequence for the approach in the last section: We can no longer switch freely between $[E]$ and $[D]$ as we did frequently in the proofs there. This phenomenon appears already in the proof of lemma 13.1.2. While the translation from powers $[D]^l$ to (products of) powers

$$[E]^{\underline{m}} := \prod_{\underline{m}=(m_j)_j} [E_j]^{m_j}$$

of reflective special divisors still works in the obvious way with the general Borcherds relations, the other direction fails in general: There may be sums of products of Heegner divisors not rationally equivalent to a self-intersection product of a toroidal boundary divisor of one-dimensional type.
Nevertheless, we can reproduce the result of proposition 13.1.4 even with more generic Borcherds relations.

Lemma 13.1.6. *Let $0 \leq l-1, m \leq n-1$ with $2 \leq l+m \leq n$. Let D be a fixed boundary divisor of $\overline{X}_\Sigma^{tor}$ of one-dimensional type. Suppose that there is a Borcherds relation of the form*

$$D \sim \frac{1}{\lambda_D}\left(\sum_{E_j} \lambda_{E_j} E_j + \sum_{D^1 \neq D} \lambda_{D^1} D^1 + \sum_{D^0} \lambda_{D^0} D^0\right),$$

with $E_j = \overline{X(K_{\lambda_j})}$ running trough all reflective special divisors and $\lambda_{E_j} \neq 0$ for every one of them. Let $\underline{m} = (m_j)_j$ be a multi-index with $|\underline{m}| = m$, then: The intersection number

$$I = [E]^{\underline{m}}[D]^l Q_{n-(l+m)}\left((1-k)c_1\left(\Omega^1_{\overline{X}}(\log\Delta)\right), \mathrm{c}\left(\Omega^1_{\overline{X}}(\log\Delta)\right)\right) \in \mathrm{CH}^n\left(X\right)$$

for $X = \overline{X}_\Sigma^{tor}$ can be written as $I = R + P$ with

i)

$$R = \sum_{E_j} \sum_{i=1}^{(n-1)-1} c_{i,j} I_{D_{\lambda_j},i,E_j,\underline{m'}},$$

where $I_{D_{\lambda_j},i,E_j,\underline{m'}}$ *is defined as the intersection number*

$$[D_{\lambda_j}]^i[E_j \cap E]^{\underline{m'}} Q_{n-(i+m)}\left((1-k)c_1\left(i_{E_j}^*\left(\Omega_X^1(\log \Delta)\right)\right), \mathrm{c}\left(\Omega_{E_j}^1(\log \Delta \cap E_j)\right)\right)$$

on E_j *with the intersection products*

$$[E_j \cap E]^{\underline{m'}} = i_{E_j}^*[E]^{m'} = \prod_{\underline{m'}=(m_s)_s} i_{E_j}^*[E_s]^{m_s}$$

and D_{λ_j} *being the boundary divisor of one-dimensional type of* $E_j = \overline{X(K_{\lambda_j})}$ *lying over the Baily-Borel cusp underlying* D*. The constants* $c_{i,j} \in \mathbb{Q}$ *depend only on the coefficients in the relation and the parameters* l, m *but not on* Σ*.*

ii)

$$P = \sum_{l+m=n} c_{\underline{m},l}[E]^{\underline{m}}[D]^l$$

is a sum of pure intersection products and the coefficients depend again only on the relation.

The proof is similar to the one of lemma 13.1.2 but more complicated:

Proof. We note that there is nothing to show if the index of the Q-polynomial is zero: The intersection product contains no logarithmic Chern classes and is of the type described in *ii*).
We can assume $m \geq 1$: If $m = 0$, then $l = l + m \geq 2$ and I is

$$\begin{aligned} & [D]^l Q_{n-(l+m)}(\dots) \\ = & \left(\sum_{E_j} \frac{\lambda_{E_j}}{\lambda_D}[E_j] + \sum_{D^1 \neq D} \frac{\lambda_{D^1}}{\lambda_D}[D^1] + \sum_{D^0} \frac{\lambda_{D^0}}{\lambda_D} D^0\right)[D]^{l-1} Q_{n-(l+m)}(\dots) \\ = & \sum_{E_j} \frac{\lambda_{E_j}}{\lambda_D}[E_j][D]^{l-1}Q_{n-(l+m)}(\dots) \end{aligned}$$

by the standard facts about $[D] \cdot [D^1]$ and $[D^0] \cdot Q_{n-(l+m)}$ for $n-(l+m) > 0$. Hence the claim follows if we can prove it for terms with $m \geq 1$.
We distinguish two cases:

(1) Let $m_j = 1$ for some j, then we can write $[E]^{\underline{m}} = [E]^{\underline{m'}}[E_j]$ with $m'_j = 0$, so

$$[E]^{\underline{m}}[D]^l Q_{n-(l+m)}(\dots) = [E]^{\underline{m'}}[E_j][D]^l Q_{n-(l+m)}(\dots)$$

which equals

$$
\begin{aligned}
= \quad & \left(i_{E_j}\right)_* \left[i^*_{E_j}\left([E]^{\underline{m'}}[D]^l\right) \cdot Q_{n-(l+m)}\left(i^*_{E_j}\left(\Omega^1_{\overline{X}}(\log\Delta)\right); \mathrm{c}\left(\Omega^1_{E_j}(\log\Delta')\right)\right)\right] \\
& - \sum_{k=1}^{n-(l+m)} \delta_{(k)}[D]^l[E]^{\underline{m'}}[E]^{k+1}Q_{n-(l+m)-k}(\ldots) \\
= \quad & \left(i_{E_j}\right)_* \left[[E_j \cap E]^{\underline{m'}}[D_{\lambda_j}]^l \cdot Q_{n-(l+m)-1}\left(i^*_{E_j}\left(\Omega^1_{\overline{X}}(\log\Delta)\right); \mathrm{c}\left(\Omega^1_{E_j}(\log\Delta')\right)\right)\right] \\
& - \sum_{k=1}^{n-(l+m)} \delta_{(k)}[D]^l[E]^{\underline{m'}}[E]^{k+1}Q_{n-(l+m)-k}(\ldots)
\end{aligned}
$$

The first term is of the claimed form, so it suffices to treat the remaining ones. If $k = n-(l+m)$, this is a pure intersection number, so we can also disregard it with respect to the claim. The remaining terms (if there are any - otherwise we are done) are of the form

$$[D]^l[E]^{\underline{m'}}[E]^{k+1}Q_{n-(l+m)-k}(\ldots) = [D]^l[E]^{\underline{m}}Q_{n-(l+m)-k}(\ldots)$$

with $k \geq 1$ and a new $\underline{m}$, so we can apply this procedure repeatedly until we arrive at an $\underline{m}$ with $m_j \geq 2$ for all non-vanishing entries.

(2) Let $m_j \geq 2$ for all non-vanishing entries of $\underline{m}$ in $[E]^{\underline{m}}[D]^lQ_{n-(l+m)}(\ldots)$. Choose any $m_i \neq 0$, then decompose $\underline{m}$ as before, so

$$[E]^{\underline{m}} = [E]^{\underline{m'}}[E_i]^{m_i}$$

without a common factor. The Borcherds relation gives

$$
\begin{aligned}
[E]^{\underline{m}} &= [E]^{\underline{m'}}[E_i]^{m_i} \\
&= [E]^{\underline{m'}}[E_i]\left(\sum_{E_j \neq E_i}\frac{\lambda_{E_j}}{\lambda_{E_i}}[E_j] + \sum_{D^1}\frac{\lambda_{D^1}}{\lambda_{E_i}}[D^1] + \sum_{D^0}\frac{\lambda_{D^0}}{\lambda_{E_i}}D^0\right)^{m_i-1} \\
&= [E]^{\underline{m'}}[E_i]\left(\sum_{|\underline{t}|=m_i-1} c_{\underline{t}}[E]^{\underline{t}} + \sum_{s=1}^{m_i-1}[D]^s c_{\underline{t}}\left(\sum_{|\underline{t}|=m_i-1-s}[E]^{\underline{t}}\right)\right. \\
&\qquad \left. + \sum_{D^1 \neq D}[D^1](\ldots) + \sum_{D^0}[D^0](\ldots)\right)
\end{aligned}
$$

for suitable $c_{\underline{t}}$ depending on the combinatorics and the Borcherds relation; note that no $[E_i]$ appears in the second factor. Substituting this expression in

$$[E]^{\underline{m}}[D]^lQ_{n-(l+m)}(\ldots)$$

we can see by the same standard arguments as before that the terms involving any $D^1 \neq D$ or any D^0 vanish.

After multiplying and reordering we see that $[E_i]$ appears exactly once, so we are in the situation described in 1) and can apply this reduction step again.

It remains to be shown that this process terminates: This follows by observing that the reduction in 1) produces either terms as in the claim or terms with decreased index of the Q-polynomial. The reduction stops once this index is zero, so the reduction stops eventually. $\square$

An explicit computation of the coefficients c_i and $c_{i,m,l}$ is desirable but remains out of reach. To accomplish this, one needs to know at least the number of reflective divisors on $X(II_{2,10}(N))$ as this is the number of summands in the appearing for example in the very first step of the reduction. Due to the lack of an Eichler criterion for $II_{2,10}(N)$ we cannot determine even this number.

Remark 13.1.7. This is actually an unconditional result in the $II_{2,10}(N)$-case: We showed the existence of a completely reflective modular form as needed in lemma 13.1.2; more precisely, we have $\lambda_{E_j} = 20 \neq 0$ for every special divisor in the relation for the Borcherds product of E_4^2/Δ considered on $II_{2,10}(N)$ as in example 12.3.8.

Of course, the case of main interest is for $m = 0$ when all of this reduces to the self-intersection products appearing in the linear term of $E(k)$.
Again, we note that

$$I_{D_{\lambda_j},1,E_j} = \chi\left(D_{\lambda_j}, \Delta \cap E_j, \Omega^n_{\overline{X}}(\log \Delta)|_{D_{\lambda_j}}^{\otimes(1-k)}\right)$$

with the notation as in lemma 13.1.6, so it can be interpreted as a logarithmic Euler characteristic on the toroidal boundary divisor D_{λ_j} of one-dimensional type of the toroidal compactification $E_j = \overline{X(K_{\lambda_j})}$.
Moreover, as in the section before, the numbers $I_{D_{\lambda_j},i,E_j} = I_{D_{\lambda_j},i,E_j,\underline{0}}$ for $i \geq 2$ are of the same form as the one we treated in the last result, so, given enough reflectively compatible Borcherds relations (on E_j, its special divisor, their special divisors and so on) we can apply this result recursively.
We just state the adapted version of corollary 13.1.5 in this case:

Proposition 13.1.8. *Fix a toroidal boundary divisor $D_0 := D \subseteq \overline{X}^{tor}_{\Sigma} =: E_0$ of one-dimensional type.*
Given any chain of lattices $\ldots \subseteq L_{i+1} \subseteq L_i \subseteq \ldots, \subseteq L = II_{2,10}(N)$ as in proposition 11.3.11 with the reflective compactifications $\overline{X(L_i)} \subseteq \overline{X}^{tor}_{\Sigma}$ of the corresponding embedded Shimura subvarieties. Suppose that for any $i = 0, \ldots, n-1$ there is a reflectively compatible Borcherds relation

$$D_i \sim \frac{1}{\lambda_{D_i}} \left(\sum_{E_i} \lambda_{E_i} E_i + \sum_{D_i^1 \neq D} \lambda_{D_i^1} D_i^1 + \sum_{D_i^0} \lambda_{D_i^0} D_i^0 \right)$$

of cycles on $\overline{X(L_i)}$, where D_i is the toroidal boundary divisor of one-dimensional type of the embedded Shimura variety $\overline{X(L_i)}$ with $\psi(D_i) \subseteq D$ under the closed immersion $\overline{X(L_i)}^{BB} \hookrightarrow \overline{X(L)}^{BB}$ (cf. section 11.3), E_i runs through the closures of the reflective

special divisor and D_i^0, D_i^1 runs through the remaining toroidal boundary divisors of zero- resp. one-dimensional type of $\overline{X(L_i)}$.
We denote such a chain $L_i \subseteq L_{i-1} \subseteq \ldots, \subseteq L = II_{2,10}(N)$ by $\underline{L}$ with $i = \operatorname{dep}(\underline{L})$ the depth *of the chain; by $E_{\underline{L}}$ we denote the toroidal compactification $\overline{X(L_i)}$ in this particular chain. If we write $[E_{\underline{L}}]^{\underline{b}}$ for some multi-index $\underline{b}$, this means*

$$[E_{\underline{L}}]^{\underline{b}} = \prod_{\underline{b}=(b_s)_s} [E_{\underline{L},s}]^{m_s},$$

where $E_{\underline{L},s}$ runs through the reflective special divisors of $E_{\underline{L}}$; the product $[D_{\underline{L}}]^{\underline{l}}$ is to be understood analogously as a product in the classes of the toroidal boundary divisors $D_{\underline{L},s}$ of $E_{\underline{L}}$.
Then: For any $1 \leq l \leq n-1$ the intersection product

$$I_{D,l,X} = [D]^l Q_{n-l}\left((1-k)c_1\left(\Omega^1_{\overline{X}}(\log \Delta)\right), \mathrm{c}\left(\Omega^1_{\overline{X}}(\log \Delta)\right)\right) \in \mathrm{CH}^n\left(X\right)$$

can be written as

$$I_{D,l,X} = R + P.$$

Here R and P are as follows:

i) The recursive *term is*

$$R = \sum_{i=0}^{n-1} \sum_{\operatorname{dep}(\underline{L})=i} c_{\underline{L}} \chi\left(D_{\underline{L}}, \Delta^{(\underline{L})}, \Omega^n_{\overline{X}}(\log \Delta)|_{D_{\underline{L}}}^{\otimes(1-k)}\right)$$

with $D^{(\underline{L})}$ the boundary divisor of one-dimensional type of $E_{\underline{L}}$ corresponding to $D_0 \subseteq \overline{X}_{\Sigma}^{tor}$, and $\Delta_{\underline{L}}$ the collection of irreducible components of the compactification divisor of $D_{\underline{L}}$. The constants $c_{\underline{L}} \in \mathbb{Q}$ depend only on the coefficients in the involved relations and the dimension, but not on k.

ii) The product term is

$$P = \sum_{i=0}^{n-1} \sum_{\substack{\text{chain } \underline{L} \\ \text{of depth } i}} \sum_{|\underline{m}|+|\underline{l}|=n-i} c_{\underline{L},\underline{m},\underline{l}} [E_{\underline{L}}]^{\underline{m}} [D_{\underline{L}}]^{\underline{l}},$$

a sum of pure intersection products and the coefficients depend again only on the relations.

Remark 13.1.9. Note that every intersection product in the terms of type ii) contains at least one toroidal boundary divisor. This follows by observing the terms without logarithmic Chern classes that can appear at each step of the reduction.

Fortunately, the existence of these completely reflective Borcherds relations is once again secured by the quasi-pullbacks of $\Psi_{E_4^2/\Delta}$ (considered on $II_{2,10}(N)$) to any of the lattices L_i as in section 12.3.
As in the conditional case before, this has an impact for the computation of the linear term:

Corollary 13.1.10. *The error term $E(k)$ is*

$$\nu_1 \sum_{i>0} \frac{(-1)^{i+1}}{2^i} \sum_{\mathrm{dep}(\underline{L})=i} c_{\underline{L}} \chi\left(D_{\underline{L}}, \Delta^{(\underline{L})}, \Omega^n_{\overline{X}}(\log \Delta)|^{\otimes(1-k)}_{D_{\underline{L}}}\right) + \mathcal{O}(1)$$

with ν_1 the number of one-dimensional cusps of $\overline{X}^{BB}$ and $c_{\underline{L}} \in \mathbb{Q}$ not depending on k.

Remark 13.1.11. Any $D_{\underline{L}}$ is a toroidal boundary component of one-dimensional type of $E_{\underline{L}}$, so it is the compactification $\overline{\mathcal{E}^{(n-2-\mathrm{dep}(\underline{L}))}}$ by closure of the Kuga-Sato variety

$$\mathcal{E}^{(n-2-\mathrm{dep}(\underline{L}))} \subseteq E_{\underline{L}} \subseteq \overline{X}^{\mathrm{tor}}_{\Sigma}.$$

The exact form of the compactification depends on the chain $\underline{L}$, that is, the position of $E_{\underline{L}}$ in $\overline{X}^{\mathrm{tor}}_{\Sigma}$.

13.2. Reduction to Euler characteristics

To determine the linear term of the error polynomial $E(k)$ we are interested in the computation of

$$\chi\left(D_{\underline{L}}, \Delta^{(\underline{L})}, \Omega^n_{\overline{X}}(\log \Delta)|^{\otimes(1-k)}_{D_{\underline{L}}}\right)$$

for non-empty $D_{\underline{L}}$, so no degeneration as in theorem 12.2.15 appears in the intersection $[D] \cdot [E_{\underline{L}}]$; otherwise we have $c_{\underline{L}} = 0$ and this term does not contribute to the error term $E(k)$.
For this section we fix a chain $L_i \subseteq L_{i-1} \subseteq \ldots, \subseteq L = II_{2,10}(N)$ of lattices of depth i and denote $l = n - 2 - \mathrm{dep}(\underline{L})$ and simply write $\overline{\mathcal{E}^{(l)}}$ for $D_{\underline{L}}$ by virtue of remark 13.1.11; moreover we use the intuitive notation $\Delta \cap \overline{\mathcal{E}^{(l)}} = \Delta^{(\underline{L})}$, so this section treats the computation of

$$\chi\left(\overline{\mathcal{E}^{(l)}}, \Delta \cap \overline{\mathcal{E}^{(l)}}, \Omega^n_{\overline{X}}(\log \Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}}\right)$$

up to terms not depending on k. By the above we implicitly assume that $\overline{\mathcal{E}^{(l)}}$ is non-empty and hence a toroidal boundary divisor of one-dimensional type of $E_{\underline{L}}$; otherwise, the Euler characteristic is trivially zero.
The first step in the further computation is the reduction to actual Euler characteristics:

Lemma 13.2.1. *We have*

$$\begin{aligned} & \chi\left(\overline{\mathcal{E}^{(l)}}, \Delta \cap \overline{\mathcal{E}^{(l)}}, \Omega^n_{\overline{X}}(\log \Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}}\right) \\ = \; & \chi\left(\overline{\mathcal{E}^{(l)}}, \Omega^n_{\overline{X}}(\log \Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}}\right) - \sum_{|\underline{b}|=l+1} \delta_{\underline{b}} D^{\underline{b}} \end{aligned}$$

where δ_l is as in proposition 4.3.2 and the elements D_i of $\Delta \cap \overline{\mathcal{E}^{(l)}}$ are the toric divisors in the boundary of $\overline{\mathcal{E}^{(l)}}$.

Proof. Observe that the elements of $\Delta \cap \overline{\mathcal{E}^{(l)}}$ are smooth and compact toric varieties by proposition 9.1.4. We apply proposition 4.3.2 to the difference of the Euler characteristic and its logarithmic counterpart: This difference is given by

$$-\sum_{|\underline{b}|\geq 1} \delta_{\underline{b}} D^{\underline{b}} Q_{l+1-|\underline{b}|}\left(c_1\left(\Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}}\right); c_t\left(\Omega^1_{\overline{\mathcal{E}^{(l)}}}(\log\Delta \cap \overline{\mathcal{E}^{(l)}}\right)\right),$$

but, as seen before in example 4.1.23, any product containing a D as well as a logarithmic Chern class vanishes, so the only remaining terms are of the claimed form. □

Again, the pure intersection numbers

$$\sum_{|\underline{b}|=l} \delta_{\underline{b}} D^{\underline{b}}$$

and the coefficients c_m appearing in lemma 13.1.2 are independent of k and hence contribute to the constant term of the dimension formula, while only the terms involving $\chi\left(\overline{\mathcal{E}^{(l)}}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}}\right)$ impact the linear term.
We will treat the computation of the pure intersection terms in section 14.1.1 and focus for the remainder of this chapter on the computation of the Euler characteristics

$$\chi\left(\overline{\mathcal{E}^{(l)}}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}}\right).$$

Reduction to modular curves

Recall the existence of the surjective morphism $\overline{X}^{\mathrm{tor}}_{\Sigma} \to \overline{X}^{\mathrm{BB}}$ from the toroidal compactification to the Baily-Borel compactification, sending toroidal boundary components to the underlying Baily-Borel cusps. Since $\overline{\mathcal{E}^{(l)}} \subseteq \overline{X}^{\mathrm{tor}}_{\Sigma}$ this restricts to a morphism

$$\pi_l : \overline{\mathcal{E}^{(l)}} \to \overline{Y(N)} = \overline{\Gamma(N)\backslash\mathbb{H}}$$

to the closure of the one-dimensional Baily-Borel cusp $Y(N) = \Gamma(N)\backslash\mathbb{H}$ that is a modular curve of full level N by proposition 6.2.4.
We will prove the following result by a strategy presented by Fiori in the aforementioned seminar talk at the Algebraic Geometry Seminar at Queen's University [2]:

Proposition 13.2.2. *For $0 \leq l \leq n-2$ the Euler characteristic*

$$\chi\left(\overline{\mathcal{E}^{(l)}}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}}\right)$$

can be expressed as Euler characteristics of bundles on $\overline{Y(N)}$:

$$\chi\left(\overline{\mathcal{E}^{(l)}}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}}\right) = \sum_{i=0}^{l}(-1)^i\binom{l}{i}\chi\left(\overline{Y(N)}, M_{2-2(k+i)}(\Gamma(N))\right).$$

[2] see again the latter part of `https://www.cs.uleth.ca/~fiori/Docs/DimFormulaTalk.pdf`

For readability we break this computation down to several steps and denote

$$\chi_l = \chi\left(\overline{\mathcal{E}^{(l)}}, \Omega^n_{\overline{X}}(\log \Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}}\right).$$

We recall: Lemma 4.2.3 tells us that

$$\left(\Omega^n_{\overline{X}}(\log \Delta)\right)^{\otimes(1-k)} = \pi_l^*(\mathcal{O}(1-k)).$$

Here $\mathcal{O}(1-k)$ is the bundle of weight $1-k$ modular forms on $\overline{X}^{\mathrm{BB}}$ (transforming with the $(1-k)$-th power of the canonical factor of automorphy), so its restriction to $\overline{\mathcal{E}^{(l)}}$ is just the bundle of weight $2-2k$ modular forms on the boundary curve $\overline{Y(N)} = \overline{\Gamma(N)\backslash\mathbb{H}}$:

$$\Omega^n_{\overline{X}}(\log \Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}} = \pi_l^*\left(\mathcal{O}(1-k)\right)|_{\overline{\mathcal{E}^{(l)}}} = \pi_l^*\left(\mathcal{O}(1-k)|_{\overline{Y(N)}}\right) = \pi_l^*(M_{2-2k}(N)).$$

For the last equation, note that sections of the k-th power of the canonical bundle on $\overline{Y(N)}$ correspond to modular form of weight $2k$ for $\Gamma(N)$. In total this yields

$$\chi_l = \chi\left(\overline{\mathcal{E}^{(l)}}, \pi_l^*(M_{2-2k}(\Gamma))\right)$$

where we shortened $\Gamma := \Gamma(N)$ as N is fixed.
We can transfer Euler characteristics between $\overline{\mathcal{E}^{(l)}}$ and $\overline{Y(N)}$ by using higher direct images:

Lemma 13.2.3. *For every $l \geq 0$ we have*

$$\chi_l = \chi\left(\overline{\mathcal{E}^{(l)}}, \pi_l^*\left(M_{2-2k}(\Gamma)\right)\right) = \sum_{i=0}^{l}(-1)^i \chi\left(\overline{Y(N)}, R^i\pi_{l*}\pi_l^*\left(M_{2-2k}(\Gamma)\right)\right)$$

This follows immediately from the corresponding general statement in proposition 4.1.26. We note that the higher direct images vanish if the index is larger then the dimension of the fibers which is $(l+1)-1 = l$.
A direct corollary of the projection formula in proposition 4.1.27 for $M_{2-2k}(\Gamma(N))$ is:

Lemma 13.2.4. *For any $i \geq 0$ we have*

$$R^i\pi_{l*}\left(\mathcal{O}_{\overline{\mathcal{E}^{(l)}}} \otimes \pi_l^*\left(M_{2-2k}(\Gamma)\right)\right) \cong M_{2-2k}(\Gamma) \otimes R^i\pi_{l*}\left(\mathcal{O}_{\overline{\mathcal{E}^{(l)}}}\right).$$

Applying this to our case gives

$$\chi_l = \sum_{i=0}^{l}(-1)^i \chi\left(\overline{Y(N)}, M_{2-2k}(\Gamma) \otimes R^i\pi_{l*}\left(\mathcal{O}_{\overline{\mathcal{E}^{(l)}}}\right)\right).$$

Unfortunately, the higher direct images $R^i\pi_{l*}\left(\mathcal{O}_{\overline{\mathcal{E}^{(l)}}}\right)$ of $\mathcal{O}_{\overline{\mathcal{E}^{(l)}}}$ are still rather complicated. We remember from section 9.1 that $\overline{\mathcal{E}^{(l)}}$ is a compactification of the l-fold fiber product of the universal elliptic curve $\mathcal{E}$ over $\overline{Y(N)}$. This is closely related to the l-fold fiber product $\overline{\mathcal{E}}^{(l)}$ of the compactification $\overline{\mathcal{E}}$ of the universal elliptic curve $\mathcal{E}$ over $Y(N)$. The following result shows that we are allowed to 'switch' compactification and fiber product:

Lemma 13.2.5. *For all $i \geq 0$ we have*

$$R^i \pi_{l*}\left(\mathcal{O}_{\overline{\mathcal{E}^{(l)}}}\right) \cong R^i \pi_{l*}\left(\mathcal{O}_{\overline{\mathcal{E}}^{(l)}}\right).$$

Proof. Note that $\overline{\mathcal{E}^{(l)}}$ and $\overline{\mathcal{E}}^{(l)}$ both have rational singularities: The first one is smooth, the second one has only toric singularities, see their description in [Del71a, Lemme 5.5] or [Sch90], which are rational.
Moreover, these schemes can be dominated by a smooth scheme Z with proper birational morphisms $Z \to \overline{\mathcal{E}^{(l)}}$ resp. $Z \to \overline{\mathcal{E}}^{(l)}$ by proposition 3.3.7: Let Σ_0 be the smooth common refinement of the induced rational polyhedral cone decomposition Σ and Σ^n_{ell}. Firstly we note that this makes sense: Since the cone decompositions $\Sigma(F)$ defining the partial compactifications over the cusps are obtained by translation of the decomposition at a given cusp, the compactification $\overline{\mathcal{E}^{(l)}}$ is of the form being treated in section 3.3. We note now that any cone of Σ can be obtained from any fixed given cone in Σ by means of the action of Λ which is properly discontinuous. Moreover, noting the special appearance of the rays in Σ^n_{ell} (cf. example 2.2.4), any cone of Σ^n_{ell} contains only finitely many rays of Σ (as their coordinates are half-integers, cf. example 11.1.12), so the number of cones in the common refinement Σ_0 lying in a given cone $\sigma \in \Sigma^n_{\text{ell}}$ is finite as well. The claim follows now from proposition 4.1.32. □

This leaves us with

$$\chi_l = \sum_{i=0}^{l} (-1)^i \chi\left(\overline{Y(N)}, M_{2-2k}(\Gamma) \otimes R^i \pi_{l*}\left(\mathcal{O}_{\overline{\mathcal{E}}^{(l)}}\right)\right)$$

and is much easier to handle as $\overline{\mathcal{E}}^{(l)}$ has the additional fiber product structure and hence allows a certain kind of factorization.
We have the following consequence of the Künneth theorem:

Lemma 13.2.6. *For $k, l \geq 1$ the k-th higher direct image $R^k \pi_{l*}\mathcal{O}_{\overline{\mathcal{E}}^{(l)}}$ is isomorphic to a direct sum of powers of $R^1\pi_{1*}\mathcal{O}_{\overline{\mathcal{E}}}$:*

$$R^k \pi_{l*}\mathcal{O}_{\overline{\mathcal{E}}^{(l)}} \cong \left(\left(R^1 \pi_{1*}\mathcal{O}_{\overline{\mathcal{E}}}\right)^{\otimes k}\right)^{\binom{l}{k}}.$$

Proof. This is certainly true for $k = l = 1$. We will use induction over $k + l = n$. We note the following two facts:

(i) Applied to the case in question with $\mathcal{F} = \mathcal{O}_{\overline{\mathcal{E}}}$ and $\mathcal{G} = \mathcal{O}_{\overline{\mathcal{E}}^{(l-1)}}$, the Künneth formula of lemma 4.1.30 is

$$R^k \pi_{l*}\left(\mathcal{O}_{\overline{\mathcal{E}}^{(l)}}\right) \cong \bigoplus_{i=0}^{k} \left(R^i \pi_1 * (\mathcal{O}_{\overline{\mathcal{E}}}) \otimes_{\mathcal{O}_{\overline{Y(N)}}} R^{k-i} \pi_{l-1*}\left(\mathcal{O}_{\overline{\mathcal{E}}^{(l-1)}}\right)\right).$$

(ii) We have $R^j\pi_{l*}\left(\mathcal{O}_{\overline{\mathcal{E}}^{(l)}}\right) = 0$ for j greater than the dimension of the fibers of π_l, that is $j > l$. In particular

$$R^j\pi_{1*}\left(\mathcal{O}_{\overline{\mathcal{E}}}\right) = 0$$

for $j \geq 2$.

Let $k + l = n + 1$, then, by (i) and (ii)

$$\begin{aligned}
R^k\pi_{l*}\mathcal{O}_{\overline{\mathcal{E}}^{(l)}} &\cong \bigoplus_{i=0}^{k}\left(R^i\pi_1 * \left(\mathcal{O}_{\overline{\mathcal{E}}}\right) \otimes_{\mathcal{O}_{\overline{Y(N)}}} R^{k-i}\pi_{l-1*}\left(\mathcal{O}_{\overline{\mathcal{E}}^{(l-1)}}\right)\right) \\
&\cong R^k\pi_{l-1*}\left(\mathcal{O}_{\overline{\mathcal{E}}^{(l-1)}}\right) \oplus \left(R^1\pi_{1*}\left(\mathcal{O}_{\overline{\mathcal{E}}}\right) \otimes_{\mathcal{O}_{\overline{Y(N)}}} R^{k-1}\pi_{l-1*}\left(\mathcal{O}_{\overline{\mathcal{E}}^{(l-1)}}\right)\right) \\
&\cong \left(\left(R^1\pi_{1*}\left(\mathcal{O}_{\overline{\mathcal{E}}}\right)\right)^{\otimes k}\right)^{\binom{l-1}{k}} \\
&\quad \oplus \left(R^1\pi_{1*}\left(\mathcal{O}_{\overline{\mathcal{E}}}\right) \otimes_{\mathcal{O}_{\overline{Y(N)}}} \left(\left(R^1\pi_{1*}\left(\mathcal{O}_{\overline{\mathcal{E}}}\right)\right)^{\otimes k-1}\right)^{\binom{l-1}{k-1}}\right) \\
&\cong \left(\left(R^1\pi_{1*}\left(\mathcal{O}_{\overline{\mathcal{E}}}\right)\right)^{\otimes k}\right)^{\binom{l-1}{k-1}+\binom{l-1}{k}} \\
&\cong \left(\left(R^1\pi_{1*}\left(\mathcal{O}_{\overline{\mathcal{E}}}\right)\right)^{\otimes k}\right)^{\binom{l}{k}}
\end{aligned}$$

by induction hypothesis and $R^0\pi_{1*}(\mathcal{O}_{\overline{\mathcal{E}}}) = \pi_{1*}\mathcal{O}_{\overline{\mathcal{E}}} = \mathcal{O}_{\overline{Y(N)}}$. □

In total, this leaves us with

$$\chi_l = \sum_{i=0}^{l+1}(-1)^i\binom{l}{i}\chi\left(\overline{Y(N)}, M_{2-2k}(\Gamma) \otimes R^1\pi_{1*}\left(\mathcal{O}_{\overline{\mathcal{E}}}\right)^{\otimes i}\right).$$

We still need to simplify $R^1\pi_*\mathcal{O}_{\overline{\mathcal{E}}}$. A direct application of lemma 4.1.33 (i.e. Serre duality in families) yields

Lemma 13.2.7. *We have*

$$R^1\pi_*\mathcal{O}_{\overline{\mathcal{E}}} \cong \left(R^0\pi_*\left(\Omega_{\overline{\mathcal{E}}/\overline{Y(N)}}\right)\right)^\vee$$

with $\Omega_{\overline{\mathcal{E}}/\overline{Y(N)}}$ *the relative cotangent sheaf of the family* $\pi : \overline{\mathcal{E}} \to \overline{Y(N)}$.

By lemma 3.3.5 and smoothness, the sheaf

$$R^0\pi_*\left(\Omega_{\overline{\mathcal{E}}/\overline{Y(N)}}\right) = \pi_*\left(\Omega_{\overline{\mathcal{E}}/\overline{Y(N)}}\right) = \Omega_{\overline{Y(N)}}$$

is the bundle $M_2(\Gamma(N))$ of modular forms of weight 2 for $\Gamma(N)$, so

$$\left(R^0\pi_*\left(\Omega_{\overline{\mathcal{E}}/\overline{Y(N)}}\right)^\vee\right)^{\otimes i} = M_{-2}(\Gamma)^{\otimes i} = M_{-2i}(\Gamma(N))$$

and hence

$$\begin{aligned}\chi_l &= \sum_{i=0}^{l+1}(-1)^i\binom{l}{i}\chi\left(\overline{Y(N)}, M_{2-2k}(\Gamma(N))\otimes R^i\pi_{l*}\left(\mathcal{O}_{\overline{\mathcal{E}}^{(l)}}\right)\right)\\ &= \sum_{i=0}^{l+1}(-1)^i\binom{l}{i}\chi\left(\overline{Y(N)}, M_{2-2(k+i)}(\Gamma(N))\right)\end{aligned}$$

as claimed in proposition 13.2.2.

This is a remarkable result: The closure $\overline{Y(N)}$ is canonical and independent of the chain $\underline{L}$ that determines the embedding. Remember that χ_l encodes the contribution of a chain $\underline{L}$ to the linear term of the error term $E(k)$ in the dimension formula, so the preceding result shows that this contribution also does not depend on $\underline{L}$ and hence every contribution of this type is equal.

This is another retrospective validation of the localization argument in the beginning of this chapter.

Further reduction

The sum of Euler characteristics as in proposition 13.2.2 could be computed by well-known tools, but there is another observation by Fiori:

Since

$$\chi\left(\overline{Y(N)}, M_{2-2(k+i)}(\Gamma_1)\right) = \chi\left(\overline{Y(N)}, M_1(\Gamma_1)^{\otimes(2-2(k+i))}\right)$$

is a linear polynomial in i, there is a lot of cancellation in the alternating sum: Recall that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P(k) = 0 = \sum_{k=0}^{n}(-1)^k\binom{n}{k}k$$

for an arbitrary polynomial P and $n > \deg(P)$, so, for $a, b \in \mathbb{Q}$ with

$$\chi\left(\overline{Y(N)}, M_{2-2(k+i)}(\Gamma)\right) = ai + b,$$

we get

$$\begin{aligned}\chi\left(\overline{\mathcal{E}^{(l)}}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}}\right) &= \sum_{i=0}^{l}(-1)^i\binom{l}{i}\chi\left(\overline{Y(N)}, M_{2-2(k+i)}(\Gamma)\right)\\ &= a\left(\sum_{i=0}^{l}(-1)^l\binom{l}{i}i\right) + b\left(\sum_{i=0}^{l}(-1)^l\binom{l}{i}\right)\\ &= \begin{cases}0 & l\geq 2\\ -a & l=1\\ b & l=0\end{cases}.\end{aligned}$$

In particular: All contributions of the embedded $\overline{\mathcal{E}^{(l)}}$ to $E(k)$ except for $l \in \{0,1\}$ vanish. It remains to compute $a, b, \in \mathbb{Q}$, i.e. the Euler characteristic of the bundle of modular forms. This is well known and may be found in [DS06]:

Proposition 13.2.8. *Let $\Gamma(N)$ be the principal congruence subgroup of level $N > 3$ and k an integer. We have*

$$\begin{aligned} & \chi\left(\overline{Y(N)}, M_{2-2(k+i)}(\Gamma(N))\right) \\ = \quad & (1-(k+i))\,(2g-2) + (1-(k+i))\nu_\infty. \end{aligned}$$

with g the genus of $Y(N)$ and ν_∞ its number of cusps.

Proof. The Riemann-Roch theorem (cf. corollary 4.1.16) states

$$\chi\left(\overline{Y(N)}, M_{2k}(\Gamma)\right) = \deg(D) + (1-g)$$

for g the genus of the modular curve $Y(N)$ and D the divisor corresponding to the bundle of weight $2k$-modular forms. The degree of the divisor of the line bundle of modular forms is

$$\deg(D) = k(2g-2) + k\nu_\infty + \lfloor k/2 \rfloor\, \epsilon_2 + \lfloor 2k/3 \rfloor\, \epsilon_3$$

with ν_∞ the number of cusps of $Y(p)$ and ϵ_i the number of elliptic points of period i. The latter two vanish due to the neatness (i.e. no non-trivial finite-order elements in $\Gamma(N)$) for $N > 3$. Taken altogether, these yield the claimed result. □

This can be made more explicit for $N = p \geq 5$ prime where the genus is known to be

$$g = \frac{(p+2)(p-3)(p-5)}{24}$$

and

$$\nu_\infty = \frac{p^2-1}{2}:$$

Corollary 13.2.9. *Let $\Gamma(p)$ be the principal congruence subgroup of prime level $p > 3$ and k an integer. We have*

$$\begin{aligned} & \chi\left(\overline{Y(p)}, M_{2-2(k+i)}(\Gamma(p))\right) \\ = \quad & (1-(k+i))\left(\left(\frac{(p+2)(p-3)(p-5)}{12}\right) - 2\right) + (1-(k+i))\frac{p^2-1}{2}. \end{aligned}$$

In view of proposition 13.2.8, the linear term $-a = -a(k)$ is just

$$-a = -a(k) = 2 - 2g + \nu_\infty$$

for g the genus of $Y(p)$ and ν_∞ the number of cusps of $Y(p)$, while

$$b = b(k) = (1-k)(2g-2+\nu_\infty) + (1-g).$$

Example 13.2.10. For $p = 19$, we have $-a(k) = -210$ and $b(k) = 210k - 405$.

We summarize the results of this chapter:

Proposition 13.2.11. *Let $\underline{L}$ be a chain $L_i \subseteq L_{i-1} \subseteq \ldots, \subseteq L = II_{2,10}(N)$ of depth i, cut out by roots of L. Let $l = n-2-i$. Denote by $\overline{\mathcal{E}^{(l)}}$ a toroidal boundary component of one-dimensional type of $E_{\underline{L}} = \overline{X(L_l)} \subseteq \overline{X}^{tor}_{\Sigma}$, then*

$$\chi\left(\overline{\mathcal{E}^{(l)}}, \Omega^n_{\overline{X}}(\log \Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}}\right)$$

$$= \begin{cases} 0 & l \geq 2 \\ 2-2g+\nu_\infty & l = 1 \\ (1-k)(2g-2+\nu_\infty)+(1-g) & l = 0 \end{cases}$$

with g the genus of $Y(p)$ and ν_∞ the number of its cusps. These can be computed as

$$g = \frac{1}{24}(p+2)(p-3)(p-5)$$

and

$$\nu_\infty = \frac{p^2-1}{2}$$

for $N = p$ prime.

For the linear coefficient of $E(k)$ only the term $(2g-2+\nu_\infty)$ matters, so we can conclude:

Theorem 13.2.12. *The linear term of the error term $E(k)$ is*

$$-\frac{\nu_1(2-2g+\nu_\infty)}{256}\left(\sum_{\mathrm{dep}(\underline{L})=8} c_{\underline{L}}\right) + \mathcal{O}(1)$$

with ν_1 the number of one-dimensional cusps of $\overline{X}^{BB}$, the genus g of $Y(N) = \Gamma(N)\backslash\mathbb{H}$, the number ν_∞ of its cusps and rational coefficients $c_{\underline{L}} \in \mathbb{Q}$ as in proposition 13.1.8, not depending on k.
For $N = p$ prime we have

$$\nu_1 = \frac{2\left[\mathrm{O}^+(II_{2,10}(p)) : \widetilde{\mathrm{SO}}^+(II_{2,10}(p))\right]}{p^{20}(1-p^{-2})},$$

$$g = \frac{1}{24}(p+2)(p-3)(p-5)$$

and

$$\nu_\infty = \frac{p^2-1}{2}.$$

If $p \equiv 3 \mod 4$, then

$$\nu_1 = 2^l p^{10}\left(p^6-1\right)\prod_{i=2}^{5}\left(p^{2i}-1\right)$$

for some $l \in \{1,2,3\}$.

To actually compute this value in an explicit way, one needs a better understanding of the coefficients $c_{\underline{L}}$, that is, of the reflectively compatible Borcherds relations and the recursive procedure in proposition 13.1.8.
We give a final example:

Example 13.2.13. For $p = 19$, the linear coefficient in $E(k)$ is

$$-2^l \left(\sum_{\mathrm{dep}(\underline{L})=8} c_{\underline{L}} \right)$$

multiplied by

$$\begin{aligned} & 151054761788784299395079214418156917125375766243360000000 \\ = \quad & 2^{11} \cdot 3^{13} \cdot 5^7 \cdot 7^7 \cdot 11 \cdot 17 \cdot 19^{10} \cdot 127^2 \cdot 151 \cdot 181^2 \cdot 911 \cdot 2251 \cdot 3833 \\ \approx \quad & 1,51 \cdot 10^{56} \end{aligned}$$

for some $l \in \{1, 2, 3\}$.

We can interpret this theorem morally as follows:
We saw before that due to Hirzebruch-Mumford proportionality, the dimension formula of $\overline{X}^{\mathrm{BB}}$ is almost governed by the geometry of the compact dual $\check{\mathcal{D}}$ of the symmetric space of $II_{2,10}(N)$; the difference between those two is mainly due to the existence of zero- and one-dimensional singularities on $\overline{X}^{\mathrm{BB}}$ which obstruct the existence of modular forms. The last result now shows that the main obstructions in this are the one-dimensional singularities.
The next chapter will treat the description and computation of the constant term of $E(k)$, finally taking into account all the terms we disregarded up until now.

14. The constant term

In the treatment of the linear term of $E(k)$ of the dimension formula in chapter 13 we made frequent use of the notation $\mathcal{O}(1)$ to subsume terms not depending on k. To get a complete dimension formula, an exact analysis of these terms is needed. This is the main objective of this chapter.
We stress again that many of the computations and results in this and the previous chapter are properties of our special choice of toroidal compactification and the special choice of special divisors, so whenever we use the notation $\overline{X}$ or $\overline{X}_\Sigma^{\mathrm{tor}}$ this denotes the reflective compactification of $X = X(\mathit{II}_{2,10}(N))$ for N sufficiently large to guarantee neatness.

Remark 14.0.1. A word of caution:
This chapter will differ in style and layout from the previous chapters which is due to the fact that there are still some of the necessary computations missing, so we will be able to give the constant term of $E(k)$ only in terms of certain geometric and combinatorial quantities instead of natural geometric properties of the space $\overline{X}_\Sigma^{\mathrm{tor}}$ or its defining features.
Unfortunately, often this will not be in the same ready-to-use manner as in most of the previous chapters: In these cases we will give algorithmic approaches to solve these problems, which reduce the tasks to extensive but straightforward computation.

As an aside: The main problem here is that most of the missing computations again depend on a given chain $\underline{L}$ of lattices defining some embedding of subschemes, whose combinatorics already was the main unsolved problem of the last chapter.
We give a list of the types of terms whose contributions to the constant terms remain to be computed:

i) pure intersection products $[D]^{\underline{b}}$ of toroidal boundary divisors on the induced compactified Kuga-Sato varieties $\overline{\mathcal{E}^{(l)}}$

ii) Euler characteristics $\chi\left(D^{\underline{b}}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{D^{\underline{b}}}\right)$ for $\underline{b}$ multiplicity free, all divisors lying over the closure of a single one-dimensional Baily-Borel cusp, but $D^{\underline{b}}$ not equal to a toroidal boundary divisor of one-dimensional type

iii) pure intersection products

$$[D]^{\underline{b}}$$

with $|\underline{b}| = n$, D_i toroidal boundary divisors of $\overline{X}_\Sigma^{\mathrm{tor}}$

iv) pure intersection products

$$[E_{\underline{L}}]^{\underline{m}}[D_{\underline{L}}]^{\underline{l}}$$

of special divisors $E_{\underline{L},s}$ and toroidal boundary divisors $D_{\underline{L},s}$ on $\overline{X(L_i)}$ with total degree $|\underline{m}|+|\underline{l}|=n-i$ for $\underline{L}$ a chain of depth i.

v) the constant coefficient of $\chi\left(\overline{\mathcal{E}^{(l)}}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}}\right)$ for $l\in\{0,1\}$, that is

$$= \begin{cases} 2-2g+\nu_\infty & l=1 \\ g-1+\nu_\infty & l=0 \end{cases}$$

for g and ν_∞ as in proposition 13.2.11.

The individual types of terms appear in various circumstances:

i) These terms appear as error terms in lemma 13.2.1: The difference between the logarithmic Euler characteristic $\chi\left(\overline{\mathcal{E}^{(l)}}, \Delta\cap\overline{\mathcal{E}^{(l)}}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}}\right)$ and the actual Euler characteristic $\chi\left(\overline{\mathcal{E}^{(l)}}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{\overline{\mathcal{E}^{(l)}}}\right)$ of the logarithmic cotangent bundle is

$$\sum_{|\underline{b}|=l+1} \delta_{\underline{b}}[D]^{\underline{b}}.$$

Note that each of these sums depends on a chain $\underline{L}$ of depth i and appears in $E(k)$ with the coefficient

$$\nu_1\frac{(-1)^{i+1}}{2^i}c_{\underline{L}}$$

where ν_1 is the number of one-dimensional Baily-Borel cusps of $X(L)$ and $c_{\underline{L}}$ the corresponding coefficient in proposition 13.1.8.

ii) The terms of this type arise as canonical contributions in the first sum of proposition 4.3.6, so the overall contribution of this to $E(k)$ is

$$\sum_{\substack{|\underline{b}|\geq 1\\ \text{multiplicity-free}}} \lambda_{\underline{b}}\chi\left(D^{\underline{b}}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{D^{\underline{b}}}\right).$$

iii) These terms arise as the constituents of the non-multiplicity-free terms

$$\sum_{\substack{|\underline{b}|\geq 1\\ \text{non-multiplicity-free}}} \lambda_{\underline{b}}[D]^{\underline{b}}$$

in proposition 4.3.6; we note that the intersection product is zero if $|\underline{b}|<n$ by lemma 10.2.1 and $\lambda_{\underline{b}}=0$ unless $b_i\geq 2$ for all non-vanishing b_i by remark 4.3.7.

iv) The terms of this type arise in the recursive process of proposition 13.1.8; the overall contribution is

$$\sum_{i=0}^{n-1} \sum_{\substack{\text{chain } \underline{L} \\ \text{of depth } i}} \sum_{|\underline{m}|+|\underline{l}|=n-i} c_{\underline{L},\underline{m},\underline{l}} [E_{\underline{L}}]^{\underline{m}} [D_{\underline{L}}]^{\underline{l}}.$$

v) These appear in proposition 13.2.11; note that this is independent of the actual chain $\underline{L}$, so the total contribution is

$$-\frac{\nu_1(2-2g+\nu_\infty)}{256}\left(\sum_{\mathrm{dep}(\underline{L})=8} c_{\underline{L}}\right) + \frac{\nu_1(g-1+\nu_\infty)}{128}\left(\sum_{\mathrm{dep}(\underline{L})=7} c_{\underline{L}}\right).$$

We remark that these contributions group naturally into classes: Cases i), ii) and v) allow an interpretation as a geometric property of a geometric natural appearing object while the cases iii) and iv) comprise of pure intersection products with possible high self-intersection. It is to be expected that we need different approaches for these two cases.
Since the contributions of the cases i), ii) and v) arise canonically and will be handled in a unified way, we call these the *canonical contributions*; the contributions iii) and iv) are the *non-canonical* ones.
Naturally, we will be able to describe and compute the canonical terms far more exact than the non-canonical ones.
Collecting all these contributions and using the formula in remark 4.3.7 for the coefficients $\lambda_{\underline{b}}$, we get the following total decomposition of the constant term:

Lemma 14.0.2. *The constant coefficient of the error term $E(k)$ is*

$$\begin{aligned}
E(0) = & \sum_{i=0}^{n-2} \sum_{\mathrm{dep}(\underline{L})=i} \nu_1 \frac{(-1)^{i+1}}{2^i} c_{\underline{L}} \sum_{|\underline{b}|=l+1} \delta_{\underline{b}} [D_{\underline{L}}]^{\underline{b}} \\
& + \sum_{\substack{\underline{b}\geq 1 \\ \textit{multiplicity-free}}} \frac{(-1)^{|\underline{b}|+1}}{2^{|\underline{b}|}} \chi\left(D^{\underline{b}}, \Omega_{X}^{n}(\log\Delta)^{\otimes(1-k)}|_{D^{\underline{b}}}\right) \\
& + \sum_{\substack{|\underline{b}|=n \\ \textit{non-multiplicity-free}}} \lambda_{\underline{b}} [D]^{\underline{b}} \\
& + \sum_{i=0}^{n-1} \sum_{\substack{\textit{chain } \underline{L} \\ \textit{of depth } i}} \sum_{|\underline{m}|+|\underline{l}|=n-i} c_{\underline{L},\underline{m},\underline{l}} [E_{\underline{L}}]^{\underline{m}} [D_{\underline{L}}]^{\underline{l}} \\
& - \frac{\nu_1(2-2g+\nu_\infty)}{256}\left(\sum_{\mathrm{dep}(\underline{L})=8} c_{\underline{L}}\right) + \frac{\nu_1(g-1+\nu_\infty)}{128}\left(\sum_{\mathrm{dep}(\underline{L})=7} c_{\underline{L}}\right).
\end{aligned}$$

For later reference we describe the k-th row of this decomposition as being of type ω for ω the roman numeral corresponding to k, in accordance with the descriptions before.

We note that we gave formulas for ν_1, ν_∞ and g in proposition 13.2.11 for $N = p$ prime, so we cannot expect a better description of the contribution of type (v) unless we get a better understanding of the coefficients $c_{\underline{L}}$.
In this chapter will explain the methods needed to compute each of the remaining terms and give some examples for these computations. To save notational effort we will drop our strict distinction between cycles and cycle classes and denote a cycle class $[D]$ simply by a representing cycle D.

14.1. Canonical contributions

We start our considerations with the canonical contributions of the constant term, more precisely of those of type i):

14.1.1. Pure intersection products on Kuga-Sato varieties

We recall the setting: Fix a chain $\underline{L}$ of lattices for the moment and use the notation $\overline{\mathcal{E}^{(l)}}$ for the toroidal boundary divisor representing $D \cdot \overline{X(L_i)}$ obtained as the intersection of a given toroidal boundary divisor D of one-dimensional type with the embedded $\overline{X(L_i)}$ and denote $l = n - 2 - \operatorname{dep}(\underline{L})$ as before. For simplicity, we denote the collection of boundary divisors of $\overline{\mathcal{E}^{(l)}}$ by Δ and its elements by D_i. We consider

$$\sum_{|\underline{b}|=l+1} \delta_{\underline{b}} D^{\underline{b}}$$

for $D_i \in \Delta \cap \overline{\mathcal{E}^{(l)}}$, with $\delta_{\underline{b}}$ the constants from proposition 4.3.2.

Computation

This expression can be characterized geometrically as follows: The Todd class $\operatorname{Td}(\mathcal{F})$ of a rank n vector bundle is a certain universal polynomial in the Chern classes $c_i = c_i(\mathcal{F})$, $1 \leq i \leq n$ of $\mathcal{F}$; we denote by $\operatorname{Td}_n(x_1, \ldots, x_n)$ the corresponding multivariate polynomial obtained by replacing c_i by x_i. Then:

$$\deg_{l+1} \operatorname{Td}(\Delta_1, \ldots, \Delta_{l+1}) = \sum_{|\underline{b}|=l+1} \delta_{\underline{b}} D^{\underline{b}} = \prod_{D \in \Delta} \frac{D}{1 - e^{-D}}$$

with Δ_i being the i-th elementary symmetric polynomial in the irreducible boundary divisors Δ.
These intersection products can be decomposed with respect to cusps:
We recall that the toroidal compactification $\overline{\mathcal{E}^{(l)}}$ is stratified by the the preimages of the Baily-Borel cusps of $\overline{\mathcal{E}^{(l)}}^{\mathrm{BB}}$ via $\pi : \overline{\mathcal{E}^{(l)}} \to \overline{\mathcal{E}^{(l)}}^{\mathrm{BB}}$ and the boundary divisors D_i obviously correspond to the non-trivial Baily-Borel cusps of $\mathcal{E}^{(l)}$. Intersection products of divisors over different cusps vanish by the standard argument via discreteness, so

$$\sum_{|\underline{b}|=l+1} \delta_{\underline{b}} [D]^{\underline{b}} = \sum_{F \text{ non-trivial cusp}} \sum_{\substack{|\underline{b}|=l+1 \\ D \in \Delta \cap \pi^{-1}(F)}} \delta_{\underline{b}} [D]^{\underline{b}}$$

and it suffices to treat the case of a fixed cusp F with the system $\Delta_F = \Delta \cap \pi^{-1}(F)$ of toroidal boundary divisors lying over F: We have to determine

$$\sum_{\substack{|\underline{b}|=l+1\\ D_i\in\Delta_F}} \delta_{\underline{b}} D^{\underline{b}} = \deg_{l+1} \mathrm{Td}(\Delta_{F,1},\dots,\Delta_{F,l+1}).$$

By the multiplicativity of the Todd classes resp. the Todd polynomial we have

$$\begin{aligned}
& \deg_{l+1} \mathrm{Td}(\Delta_{F,1},\dots,\Delta_{F,l+1}) \\
= & \sum_{k_1+k_2=l+1} \deg_{k_1} \mathrm{Td}(D_1,\dots,D_1^{l+1}) \deg_{k_2} \mathrm{Td}(\Delta'_{F,1},\dots,\Delta'_{F,l+1}) \\
= & \deg_{l+1} \mathrm{Td}(D_1,\dots,D_1^{l+1}) + \deg_1 \mathrm{Td}(D_1,\dots,D_1^{l+1}) \deg_l \mathrm{Td}(\Delta'_{F,1},\dots,\Delta'_{F,l+1}) \\
& + \sum_{k=2}^{l+1} \deg_k \mathrm{Td}(D_1,\dots,D_1^{l+1}) \deg_{l+1-k} \mathrm{Td}(\Delta'_{F,1},\dots,\Delta'_{F,l+1})
\end{aligned}$$

with $\Delta'_{F,i}$ is the i-th elementary symmetric polynomial in $D \in \Delta_F \setminus \{D_1\}$. Note that the k appearing here has nothing to do with the weight of the modular forms.
We shorten notation by removing the reference to F in Δ_F since the cusp F is fixed for the moment. Moreover, for a strictly increasing k-tuple $I = (i_1,\dots,i_k)$ (that is, $i_j < i_{j+1}$ for every j), we denote the collection $\{D_j \in \Delta | j \notin I\}$ of boundary divisors by D_{I^c} and the i-th elementary symmetric polynomial in the elements of D_{I^c} by Δ_{i,I^c}. By the inclusion-exclusion principle and the fact that

$$\deg_1 \mathrm{Td}(x_1,\dots,x_{l+1}) = \frac{1}{2}x_1$$

we can write

$$\deg_{l+1} \mathrm{Td}(\Delta_1,\dots,\Delta_{l+1})$$

as

$$\sum_{D_i} D_i^{l+1} + 2\sum_{k=1}^{l+1} \left(\frac{-1}{2}\right)^{k+1} \left(\sum_{\substack{I=(i_1,\dots,i_k)\\ \text{strictly incr.}}} \left(\prod_{j=1}^{k} \left(D_{i_j}\right)\right) \cdot \deg_{l+1-k} \mathrm{Td}\left(\Delta_{1,I^c},\dots,\Delta_{l+1,I^c}\right) \right).$$

Obviously these methods apply as well to the rest of the intersection products in the preceding equation, but our reformulation allows us to compute these in a more geometric and natural way as follows:

Proposition 14.1.1. *The intersection product*

$$\prod_{j=1}^{k} \left(D_{i_j}\right) \cdot \deg_{l+1-k} \mathrm{Td}(\Delta_{1,I^c},\dots,\Delta_{l+1,I^c})$$

for $i_1 < \dots < i_k$ is the top-degree part

$$\deg_{l+1-k} \mathrm{Td}(\Delta_{X_I})$$

of the Todd class of the (l+1-k)-dimensional toric variety $X_I := \prod_{j=1}^{k} \left(D_{i_j}\right)$.

Proof. This is a simple calculation. Note that, by the intersection theory of toric varieties in section 12.1, the product of two distinct divisors D_{ρ_1}, D_{ρ_2} is the closure $\overline{O(\sigma)}$ of the orbit corresponding to the cone $\sigma = \langle \rho_1, \rho_2 \rangle$ spanned by the generating rays of the divisor. Pulling this back to D_{ρ_1} this yields the divisor $D_{\rho'_2}$ corresponding to the ray ρ'_2 in the star $\mathrm{Star}_\Sigma(\rho_1)$ defined by ρ_2 (that is, the one-dimensional image of the projection $\sigma \to N_\mathbb{R}/\mathbb{R}\rho_1$). In other words: Products of divisors, restricted to the factors, are divisors themselves; obviously one can obtain any divisor in this way, so

$$\begin{aligned}
\prod_{j=1}^{k} \left(D_{i_j}\right) \cdot \mathrm{Td}(\Delta_{1,I^c}, \dots, \Delta_{l+1,I^c}) &= \prod_{j=1}^{k} \left(D_{i_j}\right) \cdot \prod_{\rho \in \Sigma_{I^c}(1)} \frac{D_\rho}{1 - e^{-D_\rho}} \\
&= \prod_{\rho \in \Sigma_I(1)} \frac{D_\rho}{1 - e^{-D_\rho}} \\
&= \mathrm{Td}(X_I)
\end{aligned}$$

with Σ_I the fan of the toric variety $\prod_{j=1}^{k} \left(D_{i_j}\right)$ (obtained as the star of $\langle \rho_{i_1}, \dots, \rho_{i_k} \rangle$ in Σ) by the orbit-cone correspondence. □

We noted in example 4.1.17 that the top-degree part of the Todd class of a smooth compact toric variety X is just the Euler characteristic $\chi(\mathcal{O}_X)$, so we conclude:

Corollary 14.1.2. *If $\prod_{j=1}^{k} \left(D_{i_j}\right)$ is non-empty, we have*

$$\prod_{j=1}^{k} \left(D_{i_j}\right) \cdot \deg_{l+1-k} \mathrm{Td}(\Delta_{1,I^c}, \dots, \Delta_{l+1,I^c}) = 1.$$

This reduces the computation of the intersection numbers of type (i) in lemma 14.0.2 to the following two counting problems:

1. Compute the number of zero-dimensional Baily-Borel cusps of $\mathcal{E}^{(l)}$ for any l over the modular curve $\Gamma(N)\backslash\mathbb{H}$.
2. Compute the number $C_{k,l}$ of non-vanishing intersection products $\prod_{j=1}^{k} \left(D_{i_j}\right)$ for the toric variety defining the closure of $\mathcal{E}^{(l)}$ over a given zero-dimensional cusp $F' \subset \overline{F}$.

The first of these is easily seen to be equal to the number of cusps of the modular curve $\Gamma(N)\backslash\mathbb{H}$ by the construction of $\mathcal{E}^{(l)}$ as the fiber power of the universal elliptic curve over it. For $N = p$ prime we already stated in proposition 13.2.8 that this number is

$$\frac{1}{2}\left(p^2 - 1\right).$$

In total, this yields:

Lemma 14.1.3. *For a given $\overline{\mathcal{E}^{(l)}}$ the contribution of type (i) to the constant term of $E(k)$ is*

$$\sum_{|\underline{b}|=l+1} \delta_{\underline{b}}[D]^{\underline{b}} = \frac{p^2-1}{2}\left(\sum_{D_i} D_i^{l+1} + 2\sum_{k=1}^{l+1}\left(\frac{-1}{2}\right)^{k+1} C_{k,l}\right)$$

with $C_{k,l}$ the number of non-vanishing intersection products of divisor of degree k in $\overline{\mathcal{E}^{(l)}}$ over a given cusp $F' \subseteq \pi\left(\overline{\mathcal{E}^{(l)}}\right)$.

The intersection numbers of the pure self-intersection terms D_i^{l+1} can be computed recursively as outlined in section 12.1.

Combinatorics

We sketch a possible approach to computing the numbers $C_{k,l}$:
For simplicity we restrict first to the case of $\overline{\mathcal{E}^{(n-2)}}$ and recall section 9.1. The divisors D of $\overline{\mathcal{E}^{(n-2)}}$ lying over a zero-dimensional Baily-Borel cusp F' of $\mathcal{E}^{(n-2)}$ arise as the orbits under a certain group action of the two-dimensional cones in $\Sigma(F')$ which contain as a face the isotropic ray $\rho_0 = \mathbb{R}_+ w_0$ corresponding to F. In other words: The divisors D_i are in correspondence with orbits of rays in $\mathrm{Star}_{\Sigma(F')}(\rho_0)$.
The action on cones is of the group

$$\left(\mathcal{P}(F)_{\mathbb{Z}} \cap \mathcal{P}(F')_{\mathbb{Z}}\right)/\mathcal{U}(F)_{\mathbb{Z}} \cong \Lambda \cong \mathrm{E}_8(-N)$$

where, after extending w_0 to any basis of $\Lambda \oplus \mathbb{Z}w_0$, $\lambda = (\lambda_i)_i \in \Lambda$ acts as

$$((m_i)_i, 1) \mapsto ((m_i + \lambda_i)_i, 1).$$

We can use this description to formulate a characterization of non-vanishing for products: An intersection product $\prod_{j=1}^{k}\left(D_{i_j}\right)$ of these divisors is non-trivial if and only if the representatives of the defining orbits of rays lie in a common k-dimensional cone in the star $\mathrm{Star}_{\Sigma(F')}(\rho_0)$ of ρ_0. Pulling back to $\Sigma(F')$ translates our counting problem to the following:

Lemma 14.1.4. *The number $C_{k,n-2}$ of non-vanishing intersection products $\prod_{j=1}^{k}\left(D_{i_j}\right)$ of divisors $D_i \subseteq \overline{\mathcal{E}^{(n-2)}}$ over F' is the number of Λ-orbits of $k+1$-dimensional cones containing ρ_0.*

As the decomposition $\Sigma(F')$ is already $\overline{\mathcal{P}(F')_{\mathbb{Z}}} \cong \mathrm{SO}^+(I\!I_{1,9}) \supseteq \mathrm{E}_8(-1)$-invariant, the group $\Lambda \cong \mathrm{E}_8(-N)$ acts only by the finite quotient $\mathrm{E}_8(-1)/\mathrm{E}_8(-N)$.
A good way to compute this is as follows: By construction of the reflective compactification, any $k+1$-dimensional sub-cone containing ρ_0 as a face is the image under $\mathrm{O}^+(I\!I_{1,9})$ of one of the $\binom{n-1}{k}$ cones of dimension $k+1$ of a fixed fundamental Weyl chamber that contain ρ_0. For $k=1$ their number is just given by

$$|\mathrm{Stab}_{\mathrm{O}^+(I\!I_{1,9})}(\rho_0)| = |W(\mathrm{E}_8)| = 696729600$$

as this stabilizer is just the Weyl group of E_8 by proposition 11.1.7 and the action is free on chambers. The number of orbits can now be computed by Burnside's lemma or different measures, most certainly with the help of computer algebra software.

Remark 14.1.5. We note the following for the general cases of $k \neq 1$ and $l \neq n-2$:

- For general k, this is more complicated. One may proceed as follows:
  1. Choose one of the $\binom{n-1}{k}$ cones of dimension $k+1$ of a fixed fundamental Weyl chamber, containing ρ_0 as a face.
  2. Compute the quotient of $\mathrm{O}^+(II_{1,9})$ by the Coxeter group generated by the reflection in the fundamental roots corresponding to the weights that appear as the edges of the cone in question (this is the finite quotient of $\mathrm{SO}^+(II_{1,9})$ by the stabilizer of the cone). Its order is the number of distinct $\mathrm{O}^+(II_{1,9})$-images of the chosen cone.
  3. Use Burnside's lemma to compute the number of orbits of the action of Λ on the set of cones in the last step.
- For $l \neq n-2$, the approach has to be modified slightly: We have
$$\overline{\mathcal{E}^{n-(i+2)}} \subseteq \overline{X(L_i)} \subseteq \overline{X}^{\mathrm{tor}}_{\Sigma},$$
with $L_i = \lambda_1^\perp \cap \ldots \cap \lambda_{i-1}^\perp$ so we are dealing with cones in $\Sigma_i(F) = \Sigma(F) \cap \lambda_i^\perp \cap \ldots \cap \lambda_i^\perp$ which results to restricting in step 1 to cones in $\lambda_1^\perp \cap \ldots \cap \lambda_i^\perp$ and to work inside $\mathrm{O}^+(L_i)$ in step 2. Apart from this, the algorithm sketched above works as in the case of $l = n-2$.

It is worth noting that this algorithm shows that the numbers whose computation we just sketched depend ultimately only on the structure of the cones in the Coxeter family. As the general computations of these numbers for $\overline{\mathcal{E}^l}$ for all possible l seem to involve a lot of explicit calculations, we will not approach it at the moment.
All of the previous considerations depend on the given chain $\underline{L}$ defining $\overline{\mathcal{E}^{(l)}}$ inside $\overline{X}^{\mathrm{tor}}_{\Sigma}$. To give the total contribution of type i) to the constant of the error term, we include this into the notation:

Notation 14.1.6. For a given chain $\underline{L}$ denote the number of non-vanishing intersection products of dimension k in $D_{\underline{L}} = \overline{\mathcal{E}^{n-(i+2)}} \subseteq \overline{X(L_i)}$ by $C_{k,\underline{L}}$.

Note that the second index l is no longer necessary since $l = n-2-\mathrm{dep}(\underline{L})$.

Proposition 14.1.7. *The contribution of type i) to the constant coefficient of $E(k)$ is given by*
$$\nu_1 \sum_{l=0}^{n-2} \sum_{\mathrm{dep}(\underline{L})=l} \frac{(-1)^{l+1}}{2^l} c_{\underline{L}} \frac{p^2-1}{2} \left(\sum_{D_i \in \Delta_{\underline{L}}} D_i^{l+1} + 2 \sum_{k=1}^{l+1} \left(\frac{-1}{2}\right)^{k+1} C_{k,\underline{L}} \right)$$
with ν_1 the number of one-dimensional Baily-Borel cusps and $c_{\underline{L}}, C_{k,\underline{L}} \in \mathbb{Q}$ as in proposition 13.1.8 and notation 14.1.6.
The pure self-intersection terms can be computed by way of the recursive method described in section 12.1 and the $C_{k,\underline{L}}$ may be obtained by the strategy explained in remark 14.1.5.

14.1.2. Local Euler characteristics of 0-dimensional type

We turn our attention to the contribution of type ii), namely the case of Euler characteristics

$$\chi\left(D^{\underline{b}}, \Omega^1_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{D^{\underline{b}}}\right)$$

on $D^{\underline{b}}$ for multiplicity-free $\underline{b}$ (and $D^{\underline{b}}$ not equal to a toroidal boundary divisor of one-dimensional type, as this is subsumed in the case treated in chapter 13).
As in the case before, it is useful to remember the circumstances in which these intersection products appear. They arise in the computation of the difference between the Euler characteristic and logarithmic Euler characteristic in proposition 4.3.6, so each of them comes with an extra factor of

$$\delta_{(1^{|\underline{b}|})} = -\left(\frac{-1}{2}\right)^{|\underline{b}|}.$$

Computation

We distinguish with respect to the dimension $n - |\underline{b}|$:

- If $|\underline{b}| = n$ the Euler characteristic $\chi\left(D^{\underline{b}}, \Omega^n_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{D^{\underline{b}}}\right)$ is just the regular pure intersection number of n distinct divisors and we can restrict to one of the factors, which is a smooth compact toric variety. By the intersection theory on toric varieties in section 12.1, this restricted intersection is either empty (so the intersection number is 0) or just a point with multiplicity one (so the intersection number is 1).

- If $D^{\underline{b}}$ is at least one-dimensional, we are dealing with Euler characteristics of the form

  $$\chi\left(D^{\underline{b}}, \Omega^1_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{D^{\underline{b}}}\right)$$

  for $D^{\underline{b}}$ multiplicity free and containing at most one toroidal boundary divisor of one-dimensional type, and all of the boundary divisors lying over the union of a single one-dimensional Baily-Borel cusp F with one of its boundary points F' (otherwise the product is zero for trivial reasons).

  As, by assumption, the case of $D^{\underline{b}} = D^1$ being a toroidal boundary divisor of one-dimensional type cannot occur, we can again assume that the product $D^{\underline{b}}$ contains at least one factor $\overline{O(\rho)}$ that is a divisor of the toric variety $X_{\Sigma(F')}$ lying over F'. By proposition 4.3.4 we can restrict to this divisor and compute the Euler characteristic on the toric variety $X_{\Sigma'}$ defined by the star Σ' of ρ in $\Sigma(F')$.

  We recall now example 4.1.23 to see that

  $$\chi\left(D^{\underline{b}}, \Omega^1_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{D^{\underline{b}}}\right) = 1.$$

We conclude that the Euler characteristic $\chi\left(D^{\underline{b}}, \Omega^1_{\overline{X}}(\log\Delta)^{\otimes(1-k)}|_{D^{\underline{b}}}\right)$ in either case equals 1 if and only if the intersection product $D^{\underline{b}}$ is non-empty, so it remains to count the number of non-vanishing products of toroidal boundary divisors with $|\underline{b}|$ factors.
If $D^{\underline{b}}$ contains two different toroidal boundary divisors of one-dimensional type (over necessarily differing cusps) or two toroidal boundary divisors of zero-dimensional type over differing zero-dimensional cusps, the intersection product is trivial. Moreover, the partial compactification over zero-dimensional cusps are locally isomorphic, so it suffices to restrict to the following local setting and multiply the result by ν_0, the number of zero-dimensional Baily-Borel cusps of X:
We fix Baily-Borel cusps F, F' with $F' \subseteq \partial\overline{F}$ and divisors $D_1, \dots D_{|\underline{b}|}$ over F' or F and consider the intersection

$$D^{\underline{b}} = D_1 \cdot \ldots \cdot D_{|\underline{b}|}$$

inside the toric variety $X_{\Sigma(F')}$ corresponding to the rational polyhedral decomposition $\Sigma(F')$ which is defined as in construction 11.2.1. In this setting any non-trivial intersection $D^{\underline{b}}$ corresponds to a $\overline{\mathcal{P}(F')}_{\mathbb{Z}}$-orbit of $|\underline{b}|$-codimensional cones in $\Sigma(F')$, whose number is easier to compute.
We sketch an approach for the computation in the next section.

Combinatorics

We will again put to use the special choice of the Coxeter family: The cone decompositions comprising the Coxeter family are construed by the full group $\mathrm{O}^+(II_{1,9})$, so we have an action of the quotient $\mathrm{O}^+(II_{1,9})/\widetilde{\mathrm{SO}}^+(II_{1,9}(N))$ which will be helpful.
We need to distinguish two kinds of fundamental domains of actions on $\mathcal{C}(\mathcal{F}')$: On the one hand, we have the fundamental domain $\overline{C_N}$ of the action of $\widetilde{\mathrm{SO}}^+(II_{1,9}(N))$, on the other hand we have the fundamental domain $\overline{C_0}$ of $\mathrm{O}^+(II_{1,9})$ which we used to define the Coxeter family in section 11.2. In this situation, we are interested in the number of $|b|$-codimensional cones in $\overline{C_N}$.
It is instructive to count the number of these cones in $\overline{C_0}$:

Example 14.1.8. If $|\underline{b}| = 1$, there are 9 admissible $D^{\underline{b}}$ (one for each non-isotropic ray in $\overline{C_0}$). If $|\underline{b}| \neq 1$ there are exactly $\binom{10}{|\underline{b}|}$ admissible combinations, the difference between these two cases caused by the exclusion of the case of a single toroidal boundary divisor of one-dimensional type.

To enumerate the k-dimensional cones in the fundamental domain $\overline{C_N}$, we consider them as the elements of the orbits of

$$G_{II_{1,9}(N)} := \mathrm{O}^+(II_{1,9})/\widetilde{\mathrm{SO}}^+(II_{1,9}(N))$$

of cones in the fundamental domain $\overline{C_0}$ for $\mathrm{O}^+(II_{1,9})$. The set of k-dimensional cones in $\overline{C_N}$ is partitioned into the orbits of k-dimensional cones in $\overline{C_0}$ under $G_{II_{1,9}(N)}$, so we can compute this separately for each of the latter: Fix a k-dimensional cone σ in $\overline{C_0}$. By the orbit-stabilizer theorem the number of elements in the orbit is equal to the index

$$\left[G_{II_{1,9}(N)} : \mathrm{Stab}_{G_{II_{1,9}(N)}}(\sigma)\right]$$

of the stabilizer $\mathrm{Stab}_{G_{II_{1,9}(N)}}(\sigma)$ of σ in $G_{II_{1,9}(N)}$. As the group $G_{II_{1,9}(N)}$ is finite, this is a finite computation.

Note that we computed the number of k-dimensional cones in $\overline{C_0}$ in example 14.1.8, so the number of non-trivial intersections of divisor in $\Sigma(F)$ (excluding the case of a toroidal boundary divisor of one-dimensional type) is

$$\sum_{\substack{\sigma \preceq \overline{C_0} \\ k\text{-dimensional}}} \left[G_{II_{1,9}(N)} : \mathrm{Stab}_{G_{II_{1,9}(N)}}(\sigma)\right].$$

Example 14.1.9. The number of n-dimensional cones in $\overline{X_N}$ is exactly given by the index of $\widetilde{\mathrm{SO}}^+(II_{1,9}(N))$ in $\mathrm{O}^+(II_{1,9})$ since an interior point of $\overline{C_0}$ has trivial stabilizer. By corollary 5.2.7, this number is

$$2^l p^{20} \left(p^5 - 1\right) \left(p^2 - 1\right) \left(p^4 - 1\right) \left(p^6 - 1\right) \left(p^8 - 1\right)$$

for $N = p \equiv 3 \mod 4$ prime and some $l \in \{1, 2, 3\}$.

We denote the set of k-dimensional non-totally-isotropic cones in $\overline{C_0}$ by $C^0(k)$. This definition excludes the case of the intersection product being equal to a toroidal boundary divisor of one-dimensional type.

Taking everything together this yields the following contribution to the error term:

Proposition 14.1.10. *The contribution of type ii) to the constant coefficient of $E(k)$ is given by*

$$\nu_0 \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2^k} \sum_{\sigma \in C^0(k)} \left[G_{II_{1,9}(N)} : \mathrm{Stab}_{G_{II_{1,9}(N)}}(\sigma)\right]$$

with ν_0 the number of zero-dimensional Baily-Borel cusps as in proposition 6.2.6 and

$$G_{II_{1,9}(N)} = \mathrm{O}^+(II_{1,9})/\widetilde{\mathrm{SO}}^+(II_{1,9}(N)).$$

14.2. Non-canonical contributions

We continue with the non-canonical contributions of type iii) and iv) in lemma 14.0.2. Unlike as for the canonical contributions of type i) and ii) we will not be able to give a good characterization of these in terms of geometric or combinatorial properties of $\overline{X}^{\mathrm{tor}}_{\Sigma}$, but provide guidelines for a possible computation which we will not approach here.

Note that terms of type iii) and iv) are similar in character: Both are pure intersection products of special divisors and toroidal boundary divisors on some toroidal compactification embedded in $\overline{X}^{\mathrm{tor}}_{\Sigma}$. Even though the methods for the computations are similar, we treat them separately since the contribution of type iii) does not depend on the reduction of proposition 13.1.8 while the contribution of type iv) does.

14.2.1. Pure intersection products of special divisors

We start with the contribution of type iii), so the terms of interest are of the form

$$\lambda_{\underline{b}} D^{\underline{b}}$$

for $\underline{b}$ a non-multiplicity-free multi-index, i.e. $b_j \geq 2$ for some i.

Reduction

We recall again that $\lambda_{\underline{b}} = 0$ unless $b_i \geq 2$ for all $b_i \neq 0$. Moreover, table 4.1 shows that $\lambda_{\underline{b}} = 0$ for $|\underline{b}| \leq 10$ unless $2|b_i$ for all $b_i \neq 0$.

Remark 14.2.1. This vanishing is a general phenomenon: Note that $\lambda_{\underline{b}} = \delta_{\underline{b}} \cdot s$ for some $s \in \mathbb{Q}$. The rational numbers $\delta_{\underline{b}}$ are the coefficients of $D^{\underline{b}}$ in the formal expansion of

$$\begin{aligned}\prod_{D \in \Delta} \frac{D}{1-e^{-D}} &= \prod_{D \in \Delta} \left(1 + \frac{D}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1} B_i}{(2i)!} D^{2i}\right) \\ &= \prod_{D \in \Delta} \frac{D}{1-e^{-D}} \left(1 + \frac{D}{2} + \frac{D^2}{12} + \frac{D^4}{720} + \dots\right)\end{aligned}$$

with B_i the Bernoulli numbers (cf. [Fio17] and definition 4.1.14). In particular, the coefficient $\delta_{\underline{b}}$ of any $D^{\underline{b}}$ with an odd $b_i > 1$ vanishes, so $\lambda_{\underline{b}} = 0$ as well.

Due to the standard vanishing results for intersection products on $\overline{X}^{\text{tor}}_{\Sigma}$ we can restrict to the cases of intersection numbers as in table 14.1.
In this table all but at most one of the divisors are over the same zero-dimensional Baily-Borel cusp F and the possible remaining one is a toroidal boundary divisor of one-dimensional type over a one-dimensional Baily-Borel cusp F' with $F \subseteq \overline{F'}$.
By the isomorphy of the local geometry over the cusps of the same type due to the isomorphy of the partial compactifications, it suffices to compute the intersection numbers as above for a fixed zero-dimensional cusp F and multiply accordingly:

- if D_i is of i-dimensional type in case 1), multiply the resulting intersection number by ν_i
- if all divisors are of zero-dimensional type, multiply the intersection number by ν_0
- if toroidal boundary divisors of zero- and one-dimensional type appear, multiply by
$$\left[\mathcal{P}(F)_{\mathbb{Z}} : \mathcal{P}(F)_{\mathbb{Z}} \cap \mathcal{P}(F')_{\mathbb{Z}}\right].$$
This is just the number of one-dimensional Baily-Borel cusps F_1 with $F \subseteq \overline{F_1}$.

Let F be a fixed zero-dimensional Baily-Borel cusp with cone $\mathcal{C}(F)$. As the divisors above correspond to orbits of cones in $\mathcal{C}(F)$, we can restrict to a given fundamental domain $\overline{C_N}$ of the induced action of Γ via $\widetilde{\mathrm{SO}}^+(II_{1,9}(N))$ on $\mathcal{C}(F)$ and assume that all appearing

Table 14.1.: Coefficients of pure self-intersection

intersection product $D^{\underline{b}}$	coefficient $\lambda_{\underline{b}}$
D_i^{10}	$\frac{1}{47900160}$
$D_i^8 D_j^2$	$\frac{-1}{14515200}$
$D_i^6 D_j^4$	$\frac{-1}{21772800}$
$D_i^6 D_j^2 D_k^2$	$\frac{1}{4354560}$
$D_i^4 D_j^4 D_k^2$	$\frac{1}{6220800}$
$D_i^4 D_j^2 D_k^2 D_l^2$	$\frac{-1}{1244160}$
$D_i^2 D_j^2 D_k^2 D_l^2 D_m^2$	$\frac{1}{248832}$

divisors correspond to rays in this fundamental domain. The rays in $\mathcal{C}(F)$ corresponding to the divisors are edges of a common cone in $\Sigma(F)$ since otherwise everything is trivially zero by the intersection theory in section 12.1. As several times before, we are left to count numbers of $\mathrm{O}^+(II_{1,9})/\widetilde{\mathrm{SO}}^+(II_{2,10}(N))$-orbits. By the construction of the Coxeter family we can assume that the rays defining the divisors appearing in $D^{\underline{b}}$ are rays of $\overline{C_0}$: The number of intersection products isomorphic to $D^{\underline{b}}$ is then given by

$$C(\underline{b}) := \left[\mathrm{O}^+(II_{1,9})/\mathrm{Stab}_{\widetilde{\mathrm{SO}}^+(II_{1,9}(N))}(\sigma_{\underline{b}})\right]$$

with $\sigma_{\underline{b}}$ the cone spanned by the rays ρ_i with $b_i \neq 0$.
In total, this yields:

Proposition 14.2.2. *Let F be a zero-dimensional Baily-Borel cusp of $X(II_{2,10}(N))$ and denote by $\Delta = \{D_1, \ldots, D_{10}\}$ the set of toroidal boundary divisors corresponding to the edges of the fundamental domain $\overline{C_0}$ of the natural action of $\mathrm{O}^+(II_{1,9})$ on the corresponding cone $\mathcal{C}(F)$. Assume D_1 to be the toroidal boundary divisor of one-dimensional type corresponding to the isotropic ray resp. the cusp F'.*
The contribution of type iii) to the constant coefficient of $E(k)$ is

$$\nu_0 \sum_{\underline{b} \in S} C(\underline{b}) D^{\underline{b}}$$

with S denoting the set of all multi-indices such that $D^{\underline{b}}$ is of one of the shapes in *table 14.1. Here ν_0 is the number of zero-dimensional Baily-Borel cusps and*

$$C(\underline{b}) = \left[\mathrm{O}^+(II_{1,9})/\mathrm{Stab}_{\widetilde{\mathrm{SO}}^+(II_{1,9}(N))}(\sigma_{\underline{b}})\right]$$

is the number of isomorphic images of the divisor of type $\underline{b}$, where $\sigma_{\underline{b}}$ is the cone spanned by the rays ρ_i with $b_i \neq 0$.

This result reduces the computation of the contribution of type (iii) to local computations over a given cusp. Moreover, the number of combinations to consider is considerably less than before as the set of possible divisors contains only ten elements.

Recursion

To compute the self-intersection in the last section, we use our standard practice of utilizing Borcherds relations to reduce to proper intersections and computations on subvarieties. We recall the form of the divisor of a Borcherds product on a toroidal compactification from proposition 12.3.2: It is of the form

$$\operatorname{div}(\Psi_F) = \sum_{E_i} \lambda_{E_i} E_i + \sum_{D^1} \lambda_{D^1} D^1 + \sum_{D^0} \lambda_{D^0} D^0$$

for coefficients λ as described there.
Given $D^{\underline{b}}$ as in proposition 14.2.2 and a D_i with $b_i \neq 0$, suppose we have a reflective relation

$$0 \sim \sum_{E_i} \lambda_{E_i} E_i + \sum_{D^1} \lambda_{D^1} D^1 + \sum_{D^0} \lambda_{D^0} D^0$$

with non-vanishing coefficient λ_{D_i} at D_i. We decompose $D^{\underline{b}}$ in coprime factors as usual

$$D^{\underline{b}} = D^{\underline{b}'} D_i^{b_i}.$$

We distinguish the cases of $D_i = D_1$ and $D_i \neq D_1$:

- $D_i = D_1$: We have

$$\begin{aligned} D^{\underline{b}} &= D^{\underline{b}'} D_i^{b_i} \\ &= D^{\underline{b}'} D_i^{b_i - 1} \frac{-1}{\lambda_{D_i}} \left(\sum_{E_i} \lambda_{E_i} E_i + \sum_{D^1 \neq D_1} \lambda_{D^1} D^1 + \sum_{D^0} \lambda_{D^0} D^0 \right) \\ &= \sum_{E_i} \frac{-\lambda_{E_i}}{\lambda_{D_i}} D^{\underline{b}'} D_i^{b_i - 1} E_i + \sum_{\pi(D^0) \subseteq \overline{F'}} \frac{-\lambda_{D^0}}{\lambda_{D_i}} D^{\underline{b}'} D_i^{b_i - 1} D^0 \end{aligned}$$

  by the trivial intersection product of distinct toroidal boundary divisors of one-dimensional type and the triviality of intersection products between divisors over distinct cusps.

If $b_1 - 1 = 1$ any of the intersection products in the second sum can be restricted to D_1 by the intersection theory developed in chapter 12. The divisor D_1 is a Kuga-Sato variety by proposition 9.1.2 and the remaining divisors restricted to D_1 are exactly the toroidal boundary divisors there (by the results in section 12.2), so one can compute the value of the restricted intersection numbers (even with self-intersection) on D_1 via the method outlined in section 12.1. The intersection products in the first sum are of the type appearing in the contribution of type (iv) so we defer their treatment to the next section.

If $b_i - 1 > 1$ we can apply the reflective Borcherds relation repeatedly to reduce the multiplicity of D_1 in each summand to 1. The resulting intersection products either contain only toroidal boundary divisors over F in which case they can be computed on D_1 as before, or they contain at least one special divisor, in which case we again defer to the next section.

- $D_i \neq D_1$: We have

$$\begin{aligned} D^{\underline{b}} &= D^{\underline{b}'} D_i^{b_i} \\ &= D^{\underline{b}'} D_i^{b_i - 1} \frac{-1}{\lambda_{D_i}} \left(\sum_{E_i} \lambda_{E_i} E_i + \sum_{D^1} \lambda_{D^1} D^1 + \sum_{D^0 \neq D_i} \lambda_{D^0} D^0 \right) \\ &= \sum_{E_i} \frac{-\lambda_{E_i}}{\lambda_{D_i}} D^{\underline{b}'} D_i^{b_i - 1} E_i + \sum_{F \subseteq \pi(D^1)} \frac{-\lambda_{D_i}}{\lambda_{D_i}} D^{\underline{b}'} D_i^{b_i - 1} D^1 \\ &\quad + \sum_{\pi(D^0) = F} \frac{-\lambda_{D^0}}{\lambda_{D_i}} D^{\underline{b}'} D_i^{b_i - 1} D^0 \end{aligned}$$

by analogous consideration to the case before.

If $b_i - 1 = 1$, we can restrict every summand in the second and third sum to the toric variety D_i. Analogously to before we see that the restricted divisors are toric boundary divisors, so the strategy in section 12.1 allows to compute this.

If $b_i - 1 > 1$, a repeated use of the relation yields terms with D_1 appearing without self-intersection. If such an intersection product contains no special cycle, the same argument as before allows the computation of the intersection number by section 12.1; if there is a special cycle as a factor, we once again defer to the next section.

We still need to ensure the existence of reflective Borcherds relations with non-vanishing coefficients at the toroidal boundary divisors. We computed the vanishing order of $\Psi_{E_4^2/\Delta}$ in example 12.3.3 to be 20 on every toroidal boundary divisor of one-dimensional type. Unfortunately, the expressions for the vanishing order on divisors of zero-dimensional type in proposition 12.3.2 are more complicated, see [BZ19, Theorem 5.2]. There, all the necessary ingredients except the non-canonical G_N are given, and, as noted by the authors there, a possible choice for the latter is given in [BS17, Theorem 4.2].

Due to the following fact, we can show the existence of the necessary Borcherds relations even without actually computing the vanishing order:

Lemma 14.2.3 (cf. [Pet15, Proposition 5.2.5]). *If an orthogonal modular form on $X(II_{2,10}(N))$ vanishes at every one-dimensional Baily-Borel cusp, it vanishes at order at least one on every component of the toroidal boundary.*

This is due to integral order vanishing of modular forms and the fact that every zero-dimensional cusp is a boundary point of a one-dimensional cusp, so continuity implies vanishing.
In particular:

Corollary 14.2.4. *The divisor of the reflective orthogonal modular form $\Psi(E_4^2/\Delta)$ has non-vanishing coefficients at every toroidal boundary divisor.*

This justifies the feasibility of the approach just sketched.
To summarize: The contribution of type iii) to the constant term of $E(k)$ can be expressed in terms of intersection numbers on toric varieties and Kuga-Sato varieties (which can be computed via the methods described in section 12.1), and intersection products involving reflective special cycles, which are exactly the constituents of the contribution of type iv).

14.2.2. Pure intersection products on embedded varieties

It remains to treat the case of intersection numbers appearing as in the contribution of type iv). These are, up to coefficients,

$$E_{\underline{L}}^{\underline{m}} D_{\underline{L}}^{\underline{l}}$$

for $\underline{L}$ a chain of lattices as in proposition 13.1.8 and $\underline{m}, \underline{l}$ such that $|\underline{m}| + |\underline{l}| = n - \mathrm{dep}(\underline{L})$ with $|\underline{l}| \geq 1$ (cf. chapter 13 and the process of origin of these terms in the last section). Here we denote by E_i the reflectively compatible special cycles and by D_i the boundary divisors of $X(L_{\mathrm{dep}(\underline{L})})$ and use the shorthand

$$E_{\underline{L}}^{\underline{m}} = \prod_{m=(m_i)_i} E_i^{m_i}$$

resp.

$$D_{\underline{L}}^{\underline{l}} = \prod_{l=(l_i)_i} D_i^{l_i}.$$

The computation of these works similar to the one sketched in the last section. We need the following definition: An extension of a chain $\underline{L}$ is a chain $\underline{L}'$ with $\mathrm{dep}(\underline{L}') \geq \mathrm{dep}(\underline{L})$ and $L_i' = L_i$ for all $0 \leq i \leq \mathrm{dep}(\underline{L})$.

Proposition 14.2.5. *Given a reflectively compatible Borcherds relation on $X(L_{\mathrm{dep}(\underline{L})})$ with non-vanishing coefficients at every reflective special cycle and toroidal boundary divisor for every chain $\underline{L}$ as in proposition 13.1.8, the intersection number*

$$E_{\underline{L}}^{\underline{m}} D_{\underline{L}}^{\underline{l}}$$

with $|\underline{l}| \geq 1$ can be written as a (explicitly computable) linear combination of intersection numbers of the following two types:

i) intersection numbers on a Kuga-Sato variety, realized as the toroidal boundary divisor of one-dimensional type of some $X(L_i')$ for $\underline{L'}$ an extension of $\underline{L}$

ii) intersection numbers on a toric variety, realized as the toroidal boundary divisor of zero-dimensional type of some $X(L_i')$ for $\underline{L'}$ an extension of $\underline{L}$

Proof. Given an intersection product $E_{\underline{L}}^{\underline{m}} D_{\underline{L}}^{\underline{l}}$ as in the statement one may proceed as follows:

- If the intersection product is $D^{\underline{l}}$ and one of the factors is self-intersection-free, say D_i, we can restrict to it and arrive at intersection numbers on a toric or a Kuga-Sato variety, depending on the dimensional type of D_i.
- If the intersection product is $D^{\underline{l}}$ and every boundary divisor factor has self-intersection, choose one of them, say D_j and replace it via the Borcherds relation. The resulting terms are either without special divisor factors or it contains a single special divisor factor E_i to which we can restrict as in the first step of this reduction; the latter case yields an intersection product on E_i without special divisor factors. In any case we reduced the self-intersection of D_j (in the first case) or $E_i \cap D_j$ (in the second case) without gaining special divisor factors, so repeated use reduces to one of the factors being self-intersection-free.
- If the intersection product is $E_{\underline{L}}^{\underline{m}} D_{\underline{L}}^{\underline{l}}$ with self-intersection-free $E^{\underline{m}}$ one can proceed as follows: Pull the boundary divisors inductively back to the embedded reflective $E^{\underline{m}}$ where they become boundary divisors of this toroidal compactification, so we can reduce to the case of an intersection product of the form $D^{\underline{l}}$ on some $E_{\underline{L}}$.
- If the intersection product $E_{\underline{L}}^{\underline{m}} D_{\underline{L}}^{\underline{l}}$ contains self-intersection factors in the $E^{\underline{m}}$, choose one of the factors E_i, replace $E_i^{m_i - 1}$ via the Borcherds relation and restrict everything to E_i. This reduces the maximal number of special divisors in the intersection product (now on E_i) by 1 and we can repeat this until we arrive at intersection products without self-intersection of special divisors as before.

In total this gives an algorithm to reduce general intersection products to intersection products of the stated type. □

Remark 14.2.6. Again, the existence of the needed reflectively compatible Borcherds relations with non-vanishing coefficients at all special and boundary divisors is ensured by the quasi-pullbacks of $\Psi_{E_4^2/\Delta}$ as in proposition 7.2.5.

The main point is that intersection numbers of these types can be computed by the methods developed in this thesis: The computation of arbitrary intersection numbers on Kuga-Sato varieties and toric varieties can be carried out as described in section 12.1.

The constant term

We conclude this chapter by summarizing our findings:

Theorem 14.2.7. *Fix zero- resp. one-dimensional cusp F, F' with $F \subseteq \overline{F'}$. Choose a fundamental domain $\overline{C_0}$ for the action of $\mathrm{O}^+(II_{1,9})$ on $\mathcal{C}(F)$ and denote the divisors corresponding to its rays by D_i with D_1 corresponding to the isotropic ray defining F'; denote by $C^0(l)$ the set of non-totally-isotropic l-dimensional subcones of this fundamental domain. We denote by $G_{II_{1,9}(N)}$ the finite group $\mathrm{O}^+(II_{1,9})/\widetilde{\mathrm{SO}}^+(II_{1,9}(N))$.*
The constant coefficient of the error term $E(k)$ is

$$
\begin{aligned}
&-\frac{\nu_1(2-2g+\nu_\infty)}{256}\left(\sum_{\mathrm{dep}(\underline{L})=8} c_{\underline{L}}\right)+\frac{\nu_1(g-1+\nu_\infty)}{128}\left(\sum_{\mathrm{dep}(\underline{L})=7} c_{\underline{L}}\right)\\
&+\nu_1\nu_\infty\sum_{l=0}^{n-2}\frac{(-1)^{l+1}}{2^l}\sum_{\mathrm{dep}(\underline{L})=l} c_{\underline{L}}\left(\sum_{D_i\in\Delta_{\underline{L}}} D_i^{l+1}+2\sum_{k=1}^{l+1}\left(\frac{-1}{2}\right)^{k+1}C_{k,\underline{L}}\right)\\
&+\nu_0\sum_{k=1}^{n}\frac{(-1)^{k+1}}{2^k}\sum_{\sigma\in C^0(k)}\left[G_{II_{1,9}(p)}:\mathrm{Stab}_{G_{II_{1,9}(p)}}(\sigma)\right]\\
&+\nu_0\sum_{\underline{b}\in S}\left[\mathrm{O}^+(II_{1,9}(N))/\mathrm{Stab}_{\widetilde{\mathrm{SO}}^+(II_{1,9}(N))}(\sigma_{\underline{b}})\right][D]^{\underline{b}}\\
&+\sum_{l=0}^{n-1}\sum_{\mathrm{dep}(\underline{L})=l}\sum_{|\underline{m}|+|\underline{l}|=n-l} c_{\underline{L},\underline{m},\underline{l}}[E_{\underline{L}}]^{\underline{m}}[D_{\underline{L}}]^{\underline{l}}
\end{aligned}
$$

for $n=10$, the number ν_∞ of cusps of the modular curve for the principal congruence subgroup $\Gamma(N)$ and its genus g and the number ν_i of i-dimensional Baily-Borel cusps of $X(II_{2,10}(N))$.
Here S denotes the set of all multi-indices such that $D^{\underline{b}}$ is of one of the shapes in table 14.1 and $\sigma_{\underline{b}}$ is the face of the fundamental domain $\overline{C_0}$ spanned by the rays in $\overline{C_0}$ corresponding to those D_i with $b_i \neq 0$. Moreover, $\underline{L}$ denotes a chain of strictly decreasing sublattices of $II_{2,10}(N)$ cut out by roots and $c_{\underline{L}}$ resp. $c_{\underline{L},\underline{m},\underline{l}}$ rational coefficients appearing in the recursions proposition 13.1.8. By $\Delta_{\underline{L}}$ we denote the set of boundary divisors of $X(L_{\mathrm{dep}(\underline{L})})$ and $[D_{\underline{L}}]^{\underline{B}}$ is an intersection product of its elements.
The intersection numbers of the last two terms can be computed by the methods described in section 12.1.

Note that the values of ν_0 and ν_1 can be computed by the results in example 6.2.7 if $N = p \equiv 3$ is a prime. The number ν_∞ of cusps of $Y(N)$ and its genus are well-known. The group indices appearing in this formula can be computed quite efficiently by the theory of Coxeter groups in section 11.1: The stabilizer of walls of the Tits cones are Coxeter groups themselves and hence amenable to computation.
The next and final chapter will summarize our findings about dimension formulas for $II_{2,10}(N)$, explain the remaining problems and give some ideas on how to solve them.

15. Results, Problems and Outlook

This is the final chapter of this thesis. We will use it to give an overview of the achieved results in the task of computing dimension formulas for $X(II_{2,10}(N))$, of the remaining difficulties and of possible further research directions.
The first section will summarize all of the achieved results on dimension formulas for the $II_{2,10}(N)$-lattice for $N \gg 1$ and in particular for $N = p$ prime and congruent to 3 modulo 4, before sketching the remaining problems and possible remedies. We end by stating some further natural questions.

15.1. Results

We remember theorem 10.1.3 in our case of $L = II_{2,10}(N)$ and $\Gamma = \widetilde{\mathrm{SO}}^+(II_{2,10}(N))$ neat: For $k \geq 2$ we have

$$\dim S_{10k}\left(\widetilde{\mathrm{SO}}^+(II_{2,10}(N))\right) = \mathrm{Vol}_{\mathrm{HM}}\left(\widetilde{\mathrm{SO}}^+(II_{2,10}(N))\right)\mathcal{P}(k-1) + E(k)$$

with the Hilbert polynomial $\mathcal{P}(k)$ of the compact dual

$$\check{\mathcal{D}}_{II_{2,10}(N)} = \mathrm{SO}(12)/\mathrm{SO}(2) \times \mathrm{SO}(10)$$

of $\mathcal{D}_{II_{2,10}(N)}$ and a linear error term $E(k)$.
We will make the Hirzebruch-Mumford volume and the Hilbert polynomial explicit: The work of Gritsenko, Hulek and Sankaran in [GHS07a] yields:

Proposition 15.1.1. *Let B_i be the i-th Bernoulli number. Let $m \in \mathbb{Z}, m > 0$. The Hirzebruch-Mumford volume* $\mathrm{Vol}_{\mathrm{HM}}(\tilde{\mathrm{O}}^+(II_{2,8m+2}))$ *of* $II_{2,8m+2}$ *is*

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}^+(II_{2,10})) = 2^{-4m+1}\frac{B_2 \cdot B_4 \cdot \ldots \cdot B_{8m+2}}{(8m+2)!!}\frac{B_{4m+2}}{4m+2}$$

and for a subgroup $\Gamma \subseteq \tilde{\mathrm{O}}^+(II_{2,8m+2})$ *one has*

$$\mathrm{Vol}_{\mathrm{HM}}(\Gamma) = [\mathrm{PO}^+(II_{2,8m+2}) : \mathrm{P}\Gamma]\,\mathrm{Vol}_{\mathrm{HM}}(\tilde{\mathrm{O}}^+(II_{2,8m+2})).$$

In particular, for $m = 1$, we have

$$\mathrm{Vol}_{\mathrm{HM}}(\Gamma) = [\mathrm{PO}^+(II_{2,8m+2}) : \mathrm{P}\Gamma]\frac{1}{2^5}\frac{B_2 \cdot B_4 \cdot \ldots \cdot B_{10}}{2 \cdot 4 \cdot \ldots \cdot 10}\frac{B_6}{6} = \frac{[\mathrm{P}\tilde{\mathrm{O}}^+(II_{2,10}) : \mathrm{P}\Gamma]}{92704053657600}.$$

We computed the index of $\Gamma = \widetilde{\mathrm{SO}}^+(II_{2,10}(N))$ in $\mathrm{O}(II_{2,10}) = \mathrm{O}(II_{2,10}(N))$ for $N = p$ prime congruent to 3 modulo 4 in corollary 5.2.7.
Using this result, we get:

Corollary 15.1.2. *Let $p \cong 3 \mod 4$ be a prime. The Hirzebruch-Mumford volume of the locally symmetric space with respect to $\widetilde{\mathrm{SO}}^+(II_{2,10}(p))$ is given by*

$$\mathrm{Vol}_{\mathrm{HM}}\left(\widetilde{\mathrm{SO}}^+(II_{2,10}(p))\right) = \frac{p^{30}(p^2-1)(p^4-1)(p^6-1)^2(p^8-1)(p^{10}-1)}{23176013414400}.$$

The computation of the Hilbert polynomial is easier.
Simple expansion of the known expression from proposition 10.1.5 gives:

Proposition 15.1.3. *For $n = 10$ we have*

$$\begin{aligned}\mathcal{P}(k-1) &= \binom{11k}{11k-11} - \binom{11k-2}{11k-13} \\ &= \frac{25937424601}{1814400}k^{10} + \frac{2357947691}{36288}k^9 + \frac{15648198313}{120960}k^8 + \frac{448204933}{3024}k^7 \\ &\quad + \frac{9323725543}{86400}k^6 + \frac{449171239}{8640}k^5 + \frac{6062267101}{362880}k^4 + \frac{63769541}{18144}k^3 \\ &\quad + \frac{7719437}{16800}k^2 + \frac{83963}{2520}k + 1.\end{aligned}$$

We combine all of the preceding results with our descriptions of the error term in chapter 13 and chapter 14:

Theorem 15.1.4. *Let $k \geq 2$ and $\Gamma = \widetilde{\mathrm{SO}}^+(II_{2,10}(N))$ neat. The dimension of the space $\mathcal{S}_{10k}(\Gamma)$ of cusp forms of arithmetic weight $10k$ on $X(II_{2,10}(N))$ is*

$$\frac{[\mathrm{P}\tilde{\mathrm{O}}^+(II_{2,10}) : \mathrm{P}\Gamma]}{92704053657600}\left[\binom{11k}{11k-11} - \binom{11k-2}{11k-13}\right] + E(k)$$

with $E(k)$ a linear polynomial.
If $N = p$ is prime, the error term $E(k)$ is

$$\begin{aligned}
&- \frac{\nu_1(2-2g+\nu_\infty)}{256}\left(\sum_{\mathrm{dep}(\underline{L})=8} c_{\underline{L}}\right) k \\
&- \frac{\nu_1(2-2g+\nu_\infty)}{256}\left(\sum_{\mathrm{dep}(\underline{L})=8} c_{\underline{L}}\right) + \frac{\nu_1(g-1+\nu_\infty)}{128}\left(\sum_{\mathrm{dep}(\underline{L})=7} c_{\underline{L}}\right) \\
&+ \nu_1\nu_\infty \sum_{l=0}^{n-2} \frac{(-1)^{l+1}}{2^l} \sum_{\mathrm{dep}(\underline{L})=l} c_{\underline{L}} \left(\sum_{D_i \in \Delta_{\underline{L}}} D_i^{l+1} + 2\sum_{k=1}^{l+1}\left(\frac{-1}{2}\right)^{k+1} C_{k,\underline{L}}\right) \\
&+ \nu_0 \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2^k} \sum_{\sigma \in C^0(k)} \left[G_{II_{1,9}(p)} : \mathrm{Stab}_{G_{II_{1,9}(p)}}(\sigma)\right] \\
&+ \nu_0 \sum_{\underline{b} \in S} \left[\mathrm{O}^+(II_{1,9}(N))/\mathrm{Stab}_{\widetilde{\mathrm{SO}}^+(II_{1,9}(N))}(\sigma_{\underline{b}})\right] [D]^{\underline{b}} \\
&+ \sum_{l=0}^{n-1} \sum_{\mathrm{dep}(\underline{L})=l} \sum_{|\underline{m}|+|\underline{l}|=n-l} c_{\underline{L},\underline{m},\underline{l}} [E_{\underline{L}}]^{\underline{m}} [D_{\underline{L}}]^{\underline{l}}
\end{aligned}$$

with the first summand as in theorem 13.2.12 and the remaining terms defined as in theorem 14.2.7.
The remaining unknown terms can be computed by the methods outlined in chapter 13 (for the computation of the coefficients $c_{\underline{L}}$) and chapter 14.
If $N = p \equiv 3 \mod 4$ is prime, this can be further simplified via

$$[\mathrm{P}\tilde{\mathrm{O}}^+(I\!I_{2,10}) : \mathrm{P}\Gamma] = 4p^{30}(p^2-1)(p^4-1)(p^6-1)^2(p^8-1)(p^{10}-1).$$

Even though this is by far not a complete solution of the task of determining dimension formulas for the space of cusp forms for $I\!I_{2,10}(N)$, it is a good step forward. The above theorem gives the dimension in terms of rather concrete and explicit terms. The necessary computations can be done by-and-large in an algorithmic manner. In particular, the computation involve no longer any intersection products with the hard-to-handle logarithmic Chern classes.

Problems

For convenience of the reader and reference we want to give an overview of the main difficulties and nature of the remaining open problems in the explicit computation of the preceding dimension formula:

i) To arrive at explicit results for the linear coefficient of the error term one would need an explicit understanding of the coefficients $c_{\underline{L}}$: The main problem with this is that a general Heegner divisor on $X(\Gamma)$ is no longer irreducible if $n \geq 1$ and the exact number of constituting special divisors is an important factor in the eventual result of the computations of the coefficients.

ii) A recursive process, in particular one with an unknown number of branches in each step, can easily be on the brink of computability by its pure number of steps. This may become a problem in the computation of $c_{\underline{L}}$ and $C_{k,\underline{L}}$ in (i); furthermore, this may become an issue with the intersection numbers in the last line of the constant term as the computation process in proposition 14.2.5 is highly recursive.

iii) The recursion process in proposition 14.2.5 involves the use of reflective Borcherds relations with explicitly known coefficients, so one needs to compute the vanishing order of Borcherds products on toroidal boundary divisors of zero-dimensional type as in proposition 12.3.2. This may be considerably harder than for the boundary divisors of one-dimensional type.

Problem i) may be remedied by a variant of the Eichler criterion or a suitable characterization of the irreducible components of Heegner divisor to yield explicit numbers.
A possible solution for the first part of ii) is in the work of Fiori in [Fio17, Section 6], parts of which we already used in section 4.3 and chapter 13: Given enough reflective Borcherds relations on $\overline{X}_{\Sigma}^{\mathrm{tor}}$ one can skip the recursive process of chapter 13 and reduce any self-intersection product to a proper intersection of reflective special divisors and toroidal

boundary divisors directly; even better: The coefficients of the proper intersection can be read off the Borcherds relations. This would solve the problem of computing the coefficients $c_{\underline{L}}$.
Unfortunately, reflective Borcherds products on lattices of high level seem to be rare and the search for them is computationally expensive. The obvious way is to look for vector-valued modular forms for the Weil representation of the discriminant form Δ_L; for $L = II_{2,10}(N)$ the group ring of the discriminant form alone has dimension N^{12}, which is computationally challenging and seems out of reach for all cases of our interest at the moment.
The second part of ii) and problem iii) may also profit from a wider class of reflective Borcherds relations as the number of unknown terms may decrease rapidly.
We close this thesis by giving some ideas on further lines of research and natural generalizations.

15.2. Outlook

The dimension formula in theorem 15.1.4 has two important shortcomings: The first of these is its limited scope, which is restricted to orthogonal *cusp* forms, i.e. those orthogonal modular forms vanishing at the boundary; the second one is the main prerequisite on the level N to be large enough to yield a smooth locally symmetric space $X(II_{2,10}(N))$.
Both of these are based on restrictions in the technical tools used for the Hirzebruch-Riemann-Roch approach and are by no means natural properties of orthogonal modular forms: Indeed, the definition 7.1.1 of orthogonal modular forms does not need any of these restrictions and the question of the dimension formulas makes sense without them as well. We will obviously not attempt to tackle any of these tasks here.

General orthogonal modular forms

A dimension formula for general orthogonal modular forms on a smooth $X(II_{2,10}(N))$ may be derived from the corresponding formula for orthogonal cusp forms. Remember the line bundle $\mathcal{L}$ of weight 1 orthogonal modular forms from proposition 7.1.3. The orthogonal cusp forms of weight nk are exactly the global sections of $\mathcal{L}^{\otimes nk}(-\Delta)$ for the compactification divisor $\Delta = \overline{X}^{\mathrm{tor}}_{\Sigma} \setminus X$. We have the short exact sequence

$$0 \longrightarrow \mathcal{L}^{\otimes nk}(-\Delta) \longrightarrow \mathcal{L}^{\otimes nk} \longrightarrow \mathcal{L}^{\otimes nk}|_{\Delta} \longrightarrow 0,$$

so the growth behavior of the dimension of spaces of orthogonal modular forms and orthogonal cusp forms agree. Moreover, a general orthogonal modular form may be turned into a non-trivial orthogonal cusp form by subtracting a suitable linear combination of other orthogonal modular forms such that the restriction of the resulting form to Δ becomes trivial. This would reduce the question of dimension formulas to combinatorics of components of the boundary divisors.

In the classical elliptic case this is achieved with the help of Eisenstein series which do not vanish at exactly one of the irreducible components of the compactification divisor. A similar theory of orthogonal Eisenstein series could pave the way to generalize dimension formulas from orthogonal cusp forms to the general case.

Non-neat locally symmetric spaces

The problem of finding dimension formulas for orthogonal cusp forms with respect to a non-neat subgroup Γ of $\mathrm{O}(II_{2,10})$ is more serious. The main idea, put forward by Tai in [Tai80], is to work with a smooth finite cover $X(\Gamma')$ of $X(\Gamma)$ which is induced by an intermediate finite-index neat subgroup $\Gamma' \subseteq \Gamma$ whose existence we saw in lemma 5.2.11. Note that $X(\Gamma) = \Gamma\backslash X(\Gamma')$ and $X(\Gamma')$ is smooth as in the preceding considerations. The vector space of orthogonal cusp forms $S_k(\Gamma)$ for Γ is just the set of Γ/Γ'-invariant elements in $S_k(\Gamma')$, so

$$S_k(\Gamma) = S_k(\Gamma')^{\Gamma}.$$

In order to compute the dimension of this space one is led to use some version of the *Atiyah-Bott fixed point theorem*, cf. [Tai80] attributing this particular application to Hirzebruch in [Hir66], which gives

$$\dim S_k(\Gamma) = \dim S_k(\Gamma')^{\Gamma} = \frac{1}{|\Gamma/\Gamma'|} \sum_{\gamma\in\Gamma/\Gamma'} \mathrm{tr}\left(\gamma^* | S_k(\Gamma')\right).$$

The latter traces may be expressed and computed in terms of geometric properties of the smooth $X(\Gamma')$ resp. a suitable toroidal compactifications by the Atiyah-Singer index theorem in [AS68, Theorem 4.6]. This computation seems to be a hard task by itself. A successful implementation in the Siegel case has been achieved by Tsushima in [Tsu82].

Speculation

We want to close with a bit of speculation by listing three interesting facts:

- Gritsenko and Hulek proved in [GH16] that an open subset of the orthogonal modular variety $\mathrm{O}^+(II_{2,10})\backslash\mathcal{D}_{II_{2,10}}$ represents isomorphism classes of numerically polarized Enriques surfaces in a quite natural way.
- Alexeev in [Ale02], building on earlier work of Namikawa in [Nam09], extended the moduli interpretation of the Siegel modular variety to the very natural toroidal compactification induced by the second Voronoi decomposition.
- The work of Alexeev, Engel and Thompson in [AET19] gave a moduli interpretation for a toroidal compactification of the orthogonal modular variety $X(II_{1,1} \oplus \mathrm{E}_8^2 \oplus A_1)$ for the hyperbolic lattice $II_{1,1} \oplus \mathrm{E}_8^2 \oplus A_1$ of signature $(18, 1)$; the defining admissible family there is induced by the local notion of a Coxeter fan, so it can be thought of as a form of reflective compactification.

Together, these observations call for examining the role of reflective compactifications as naturally appearing choices of toroidal compactifications of orthogonal modular varieties with complex structure in general.
A concrete further question to pursue might be the following: Is there a natural moduli interpretation of the reflective compactification of $X(II_{2,10})$ in terms of degenerations of numerically polarized Enriques surfaces?

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